A UNITED STATES DEPARTMENT OF COMMERCE PUBLICATION

SECTION B

JOURNAL OF RESEARCH

of the NATIONAL BUREAU OF STANDARDS

JNBBAU 77B(3&4) 73-214 July-Dec. 1973

Mathematical Sciences

U.S. DEPARTMENT OF COMMERCE
National Bureau of Standards

Journal of Research

of the

National Bureau of Standards

B. Mathematical Sciences

July–Dec. • 1973

VOLUME • 77B

NUMBERS • 3 & 4

Editor: Morris Newman

Associate Editor: F. W. Olver

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1973

J. Res. Nat. Bur. Stand. (U.S.), B

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The Secretary of Commerce has determined that the publication of this periodical is necessary in the transaction of the public business required by law of this Department. Use of funds for printing this periodical has been approved by the Director of the Office of Management and Budget through June 30, 1976.

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 3 and 4, July–December 1973

An Application of Schur's Lemma on Irreducible Sets of Matrices in Continuum Mechanics

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(February 2, 1973)

In a recent study of the thermodynamic restrictions of a theory of compressible, viscoelastic fluids, Fong and Simmons (ZAMP **23**, No. 5 (1972)) encountered a problem of integrating the following matrix identity:

$$\underset{=}{M}[\,\underset{=}{H}\hat{U}_{,\underset{=}{C}}(\underset{=}{H}^{T}\underset{=}{M}\underset{=}{H})\underset{=}{H}^{T} - \hat{U}_{,\underset{=}{C}}(\underset{=}{M})] - \underset{=}{H}\hat{U}_{,\underset{=}{C}}(\underset{=}{H}^{T}\underset{=}{H})\underset{=}{H}^{T} = \underset{=}{0},$$

where $\hat{U}_{,\underset{=}{C}}$ denotes the gradient of the scalar-valued function $U=\hat{U}(\boldsymbol{C})$ with respect to its matrix argument $\boldsymbol{C}$ which is symmetric and positive-definite. The identity is valid for every symmetric positive-definite $\boldsymbol{M}$ and every unimodular $\boldsymbol{H}$. The symbol $\boldsymbol{H}^T$ denotes the transpose of the matrix $\boldsymbol{H}$. The solution of the problem is presented here in detail as an example of applying, probably for the first time, Schur's lemma on irreducible sets of matrices in theoretical continuum mechanics.

Key words: Continuum mechanics; elasticity; integration; matrix calculus; matrix identity; matrix theory; reducibility; Schur's lemma; strain energy.

1. Introduction

Continuum mechanics, or the mechanics of a deformable medium, depends heavily on the use of standard results in matrix theory for the formulation of problems and their solutions. For example, a "hyperelastic material" is characterized by the following constitutive equation when the thermal variables are ignored:

$$\underset{=}{T} = \rho\underset{=}{F}\sigma_{\underset{=}{F}}(\underset{=}{F})^{T}, \qquad T^{k}_{m} = \rho F^{k}_{\alpha}\frac{\partial\sigma}{\partial F^{m}_{\alpha}}. \tag{1.1}$$

Here $\boldsymbol{T}$ is the Cauchy stress tensor with a matrix representation T^k_m, which specifies the actual contact force per unit area in the spatial coordinate system x^k, $k=1,2,3$. The symbol ρ stands for the mass density per unit volume associated with a particle at x^k. To define $\boldsymbol{F}$, the deformation gradient tensor with a matrix representation F^k_α, we need to introduce a reference configuration κ with respect to which each material particle is given a coordinate label X^α, $\alpha=1,2,3$. The deformation gradient matrix is then defined as $F^k_\alpha=\partial x^k(X^\beta)/\partial X^\alpha$. The scalar function $\sigma(\boldsymbol{F})$ is called the strain-energy function of the hyperelastic material. The symbol $\sigma_{\boldsymbol{F}}$ stands for the gradient of σ with respect to $\boldsymbol{F}$, and is, therefore, itself a matrix with its transpose denoted by $\sigma_{\boldsymbol{F}}(\boldsymbol{F})^T$. Equation (1.1) states that the response of a hyperelastic material is completely determined for a given set of values of ρ and $\boldsymbol{F}$, provided the form of the scalar function σ can be determined experimentally. As it stands, σ depends on a 3×3 matrix variable or a total of nine components of the matrix F^k_α. A combination of physical requirement (strain energy must be frame-indifferent), and a standard result in matrix theory (polar decomposition of $\boldsymbol{F}$ into a product of an orthogonal $\boldsymbol{R}$ and a symmetric $\boldsymbol{U}$) reduces the

AMS Subject Classification: 73B99; 20C15.

number of variables in the function σ from nine to six, i.e., $\sigma(\boldsymbol{F}) = \sigma(\boldsymbol{U})$. For additional examples, see, e.g., Truesdell and Noll [1],[1] Murnaghan [2], etc.

In a recent study of the thermodynamic restrictions of a theory of compressible, viscoelastic fluids, Fong and Simmons [3] encountered the problem of integrating the following matrix identity:

$$\underline{\underline{M}}\,[\underline{\underline{H}}\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\underline{\underline{H}})\underline{\underline{H}}^T - \hat{U}_{,\underline{\underline{C}}}(\underline{\underline{M}})] - \underline{\underline{H}}\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{H}})\underline{\underline{H}}^T \equiv \underline{\underline{0}}, \tag{1.2}$$

where $\hat{U}_{,\mathbf{C}}$ denotes the gradient of a scalar-valued function $U = \hat{U}(\mathbf{C})$ with respect to its matrix argument $\boldsymbol{C}$ which is, by definition, symmetric and positive-definite.[2]. The identity is valid for every symmetric, positive-definite $\boldsymbol{M}$ and every unimodular $\boldsymbol{H}$, i.e., $\det \boldsymbol{H} = 1$. It turns out that the integrability of (1.2) depends in a crucial way on two basic results in matrix theory. The purpose of this expository paper is to bring to the readers' attention these results which are well-known to mathematicians but not necessarily to workers in continuum mechanics:

Fact 1 The set of all symmetric, positive-definite matrices is irreducible.

Fact 2 If a matrix $\boldsymbol{Y}$ commutes with each matrix of an irreducible set, then $\boldsymbol{Y}$ is a scalar matrix, i.e., $\boldsymbol{Y} = \lambda \boldsymbol{I}$.

In section 2, the notion of "reducibility" of a set of matrices is first defined. A proof relating the notion of "reducibility" with that of an invariant subspace is also given. In section 3, we prove "fact 1" with a scheme of reasoning essentially due to Newman [4]. In section 4, we begin with Schur's lemma on irreducible sets of matrices and use it to prove "fact 2." The integration of (1.2) using both facts 1 and 2 is given in section 5. Finally, a discussion of the significance of the new result appears in section 6.

2. Reducibility of a Set of Matrices

We reproduce here the formal definition of the notion of "reducibility" of a set of matrices, $\mathscr{A} = \{\boldsymbol{A}_{(n\times n)}\}$, over the complex field, as presented by Newman [5]. The set $\mathscr{A}$ is said to be *reducible* if there exists a fixed, nonsingular matrix $\boldsymbol{S}_{(n\times n)}$ and fixed positive integers p, q, such that for each $\boldsymbol{A}$ in $\mathscr{A}$,

$$\underline{\underline{S}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{S}} = \begin{pmatrix} \underline{\underline{B}}_{(p\times p)} & \underline{\underline{O}} \\ \underline{\underline{D}}_{(q\times p)} & \underline{\underline{E}}_{(q\times q)} \end{pmatrix} \tag{2.1}$$

The symbol $\boldsymbol{O}$ represents a block of zeros with, of course, p rows and q columns. If no such $\boldsymbol{S}$ can be found, the set $\mathscr{A}$ is said to be *irreducible*. Examples of reducible and irreducible sets of matrices are:

Example 2.1 The set consisting of a single $n \times n$ matrix A alone, $n > 1$, is reducible.

Example 2.2 The set consisting of all 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is reducible.

Example 2.3 The set consisting of all $n \times n$ matrices of the form $\begin{pmatrix} \underline{\underline{U}} & \underline{\underline{V}} \\ \underline{\underline{V}} & \underline{\underline{U}} \end{pmatrix}$ with respect to some fixed partitioning is reducible.

Example 2.4 The set of all $n \times n$ column stochastic matrices having all column sums equal to 1 is reducible.

Example 2.5 The set $\mathscr{A} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is irreducible.

Additional examples of irreducible sets will be given in the next section. For those reducible sets given in the above examples, the reader can find the corresponding fixed matrix $\boldsymbol{S}$ in the book by

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] The scalar function $\hat{U}$, as it appears in reference [3], also depends on a scalar parameter ζ, i.e., $U = \hat{U}(\underline{\underline{C}}, \zeta)$. For our purposes here, this dependence is suppressed for brevity.

Newman [5]. We now wish to interpret the notion of reducibility by proving its equivalence to the existence of an invariant subspace:

Remark 2.1 Let V be the n-dimensional vector space over the complex field, and let $\mathscr{A}=\{\mathbf{A}_{(n\times n)}\}$ be a set of matrices which is reducible. Then there exists a subspace W in V such that W is invariant under any sequence of transformations given by the matrices in the set $\mathscr{A}$.

PROOF: Interpreting matrices in $\mathscr{A}$ as transformations, we have

$$\{\underline{\underline{A}}\,\underline{v}\}=\left\{\underline{\underline{S}}^{-1}\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{\underline{S}}\,\underline{v}\right\}, \tag{2.2}$$

valid for each $\boldsymbol{v}$ in V and each $\boldsymbol{A}$ in the reducible set $\mathscr{A}$. Consider the set of vectors of the form

$$\underline{w}=\begin{pmatrix}\underline{0}_{(p\times 1)}\\ \underline{y}_{(q\times 1)}\end{pmatrix}. \tag{2.3}$$

Then we can define a subspace W consisting of all the $\boldsymbol{w}$'s with the property that W is invariant under matrix transformations of the type

$$\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}, \quad \text{i.e.,} \quad \begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{w}=\underline{w}_1$$

is necessarily an element of W. We now wish to show that W is also invariant under the set $\mathscr{A}$. Let $\boldsymbol{S}^{-1}\boldsymbol{w}=\boldsymbol{u}$, i.e., $\boldsymbol{S}\boldsymbol{u}=\boldsymbol{w}$, and $\boldsymbol{S}\boldsymbol{u}_1=\boldsymbol{w}_1$. Then we have

$$\underline{\underline{S}}^{-1}\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{\underline{S}}\,\underline{u}=\underline{\underline{S}}^{-1}\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{D}} & \underline{\underline{E}}\end{pmatrix}\underline{w}=\underline{\underline{S}}^{-1}\,\underline{w}_1=\underline{u}_1.$$

Since a fixed transformation matrix $\boldsymbol{S}$ when applied to all the vectors in a subspace W does not alter the collection of vectors in that subspace, we conclude that the reducibility of $\mathscr{A}$ implies the existence of an invariant subspace W under $\mathscr{A}$.

Remark 2.2 Given a subspace W of the n-dimensional vector space V and given a set of matrices $\mathscr{A}=\{\boldsymbol{A}_{(n\times n)}\}$ under which W remains invariant, then the set $\mathscr{A}$ is reducible.

PROOF: Let the subspace W be of dimension q, $q<n$, and let $\boldsymbol{w}$ be any vector in W. Then there exists a linear transformation with matrix $\boldsymbol{S}$ such that every vector $\boldsymbol{w}$ can be brought to the form:

$$\underline{w}=\underline{\underline{S}}\begin{pmatrix}\underline{0}_{(p\times 1)}\\ \underline{y}_{(q\times 1)}\end{pmatrix}. \tag{2.4}$$

The condition that W is invariant under $\mathscr{A}$ implies $\{\boldsymbol{A}\boldsymbol{w}\}\subset\{\boldsymbol{w}\}$. Substituting the representation of $\boldsymbol{w}$ as given in (2.4) into the condition of invariance, we get

$$\left\{\underline{\underline{A}}\,\underline{\underline{S}}\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\}\subset\left\{\underline{\underline{S}}\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\},$$

$$\text{i.e.,}\ \left\{\underline{\underline{S}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{S}}\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\}\subset\left\{\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix}\right\}. \tag{2.5}$$

The statement (2.5) implies $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ must be of the form $\begin{pmatrix}\underline{\underline{B}} & \underline{\underline{O}}\\ \underline{\underline{Q}} & \underline{\underline{E}}\end{pmatrix}$. Hence the set $\mathscr{A}$ is reducible.

Combining remarks 2.1 and 2.2, we arrive at the following useful result:

Remark 2.3 A set of square matrices, $\mathscr{A}=\{\boldsymbol{A}_{(n\times n)}\}$, over the complex field is reducible, if, and only if, there exists an invariant subspace under $\mathscr{A}$.

3. Irreducibility of the Set of Symmetric, Positive-Definite Matrices

We now wish to use remark 2.3 to show whether a given set of matrices is reducible or not. The following remark is due to Newman [4]:

Remark 3.1 The set of matrices consisting of a diagonal matrix with nonzero and distinct eigenvalues, i.e., diag $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \neq \lambda_j$ for $i \neq j$, $1 \leq i, j \leq n$, and a special matrix $\boldsymbol{J}=(J_{ij})$, with $J_{ij}=1$, $1 \leq i, j \leq n$, is irreducible.

PROOF: Let $\boldsymbol{x}=(x_1, x_2, \ldots, x_n)^T$ be a nonzero vector in the n-dimensional vector space V. Let $\boldsymbol{D}$ be the diagonal matrix with nonzero, distinct eigenvalues, $\lambda_1, \ldots, \lambda_n$. The proof for remark 3.1 can be broken into three steps as follows:

Step 1 For $\boldsymbol{x} \neq \boldsymbol{0}$, there exists a positive integer k such that $\boldsymbol{D}^k\boldsymbol{x} \neq \boldsymbol{0}$. Suppose the statement is false, i.e., $\boldsymbol{D}^i\boldsymbol{x}=\boldsymbol{0}$ for $1 \leq i \leq n$. Then it is possible to have the system:

$$\begin{cases} \lambda_1 x_1+\lambda_2 x_2+\ldots\ldots+\lambda_n x_n=0 \\ \lambda_1^2 x_1+\lambda_2^2 x_2+\ldots\ldots+\lambda_n^2 x_n=0 \\ \qquad\ldots\ldots\ldots \\ \lambda_1^n x_1+\lambda_2^n x_2+\ldots\ldots+\lambda_n^n x_n=0. \end{cases} \tag{3.1}$$

The system (3.1) has a nontrivial solution based on the hypothesis that $\boldsymbol{x} \neq \boldsymbol{0}$. Hence the determinant must vanish, contrary to the well-known result that a determinant of the form:

$$\Delta=\begin{pmatrix} \lambda_1\lambda_2\ldots\lambda_n \\ \lambda_1^2\lambda_2^2\ldots\lambda_n^2 \\ \ldots\ldots\ldots \\ \lambda_1^n\lambda_2^n\ldots\lambda_n^n \end{pmatrix}=\lambda_1\lambda_2\ldots\lambda_n \prod_{1\leq i<j\leq n}(\lambda_j-\lambda_i)\neq 0, \qquad (\lambda_i\neq\lambda_j\neq 0,\ i\neq j)$$

is never zero. Hence for $\boldsymbol{x} \neq \boldsymbol{0}$, there exists k such that

$$\lambda_1^k x_1+\lambda_2^k x_2+\ldots+\lambda_n^k x_n \neq 0.$$

Step 2 We now calculate the vector $\boldsymbol{J}\boldsymbol{D}^k\boldsymbol{x}$ and conclude that it is equivalent to the vector $\boldsymbol{y}=(1, 1, \ldots, 1)^T$ up to a nonzero scalar multiplying constant. Let us now calculate $\boldsymbol{D}\boldsymbol{y}$, $\boldsymbol{D}^2\boldsymbol{y}$, $\ldots$, $\boldsymbol{D}^{n-1}\boldsymbol{y}$, and obtain the following set of vectors:

$$\begin{aligned} \boldsymbol{y}&=(1, 1, \ldots, 1)^T, \\ \boldsymbol{D}\boldsymbol{y}&=(\lambda_1, \lambda_2, \ldots, \lambda_n)^T, \\ \boldsymbol{D}^2\boldsymbol{y}&=(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2)^T, \\ &\ldots\ldots \\ \boldsymbol{D}^{n-1}\boldsymbol{y}&=(\lambda_1^{n-1}, \lambda_2^{n-1}, \ldots, \lambda_n^{n-1})^T. \end{aligned}$$

Since the determinant is the well-known Vandermondian which is nonzero as long as the λ_i are distinct, we conclude that the above form a linearly independent set of n vectors and span the space.

Step 3 Since the set $\mathscr{A}=\{\boldsymbol{D}, \boldsymbol{J}\}$ of matrices when applied to a nonzero vector $\boldsymbol{x}$ generates the entire space, there is no proper subspace invariant under $\mathscr{A}$. Hence, by remark 2.3, the set $\mathscr{A}$ is irreducible.

Since both matrices $\boldsymbol{D}$ and $\boldsymbol{J}$ as defined in remark 3.1 are symmetric, it is trivial to conclude that,

Remark 3.2 The set of all symmetric matrices of any order over the complex field is irreducible.

Since any symmetric matrix can be made into a positive-definite symmetric one by the addition of a scalar matrix $\alpha \boldsymbol{I}$, where α is any nonzero scalar, and $\boldsymbol{I}$ is the identity matrix, and it is easy to show that such an addition does not affect the property of reducibility of a given set of symmetric matrices, we conclude that,

Remark 3.3 The set of all symmetric, positive-definite matrices of any order over the complex field is irreducibile. (This was stated earlier as "*Fact 1.*")

4. Shur's Lemma on Irreducible Sets of Matrices

We reproduce here the celebrated Schur's Lemma as stated in [5], p. 3:

THEOREM (Schur's Lemma): *Let* $\mathscr{A}=\{\boldsymbol{A}\}$, $\mathscr{B}=\{\boldsymbol{B}\}$ *be irreducible sets of* $n\times n$ *matrices,* $m\times m$ *matrices respectively. Let* $\boldsymbol{M}$ *be a fixed* $m\times n$ *matrix which determines a* $1-1$ *correspondence between* $\mathscr{A}$ *and* $\mathscr{B}$ *such that* $\boldsymbol{MA}=\boldsymbol{BM}$. *Then either* $\boldsymbol{M}=\boldsymbol{0}$, *or* $m=n$ *and* $\boldsymbol{M}$ *is nonsingular.*

The proof for the above theorem is given in [5] and is omitted here for brevity. It is, however, instructive to repeat the proof for an important corollary as follows:

COROLLARY: *If a matrix* $\boldsymbol{Y}$ *commutes with each matrix of an irreducible set* $\mathscr{A}$, *then* $\boldsymbol{Y}$ *is a scalar matrix, i.e.,* $\boldsymbol{Y}=\lambda \boldsymbol{I}$.

PROOF: Let λ be any eigenvalue of $\boldsymbol{Y}$. Then $\boldsymbol{Y}-\lambda \boldsymbol{I}$ is singular. It is easy to see that the matrix $\boldsymbol{Y}-\lambda \boldsymbol{I}$ also commutes with each matrix of $\mathscr{A}$. Schur's lemma now implies that $\boldsymbol{Y}-\lambda \boldsymbol{I}$ must be $\boldsymbol{0}$. Hence $\boldsymbol{Y}=\lambda \boldsymbol{I}$.

This corollary was referred to earlier in the introduction as "*Fact 2.*"

5. Integration of the Matrix Identity (1.2)

Let us rewrite (1.2) by introducing $\boldsymbol{Y}=\boldsymbol{H}\hat{U}_{,\underline{\underline{C}}}(\boldsymbol{H}^T\boldsymbol{H})\boldsymbol{H}^T$:

$$\underline{\underline{H}}\,\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\underline{\underline{H}})\underline{\underline{H}}^T-\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{M}})=\underline{\underline{M}}^{-1}\underline{\underline{Y}}. \tag{5.1}$$

Since $\hat{U}$ depends on a symmetric argument, the gradient $\hat{U}_{,\boldsymbol{C}}$ is necessarily symmetric. This implies $\boldsymbol{Y}$ is symmetric as well as the left-hand side of (5.1). Since $\boldsymbol{M}$ is symmetric and positive-definite, $\boldsymbol{M}^{-1}$ is also symmetric and positive-definite. The identity (5.1) tells us that the product of two symmetric matrices, $\boldsymbol{M}^{-1}$ and $\boldsymbol{Y}$ is also symmetric. A standard result in matrix theory says that the necessary and sufficient condition for the product of two symmetric matrices to be again symmetric is that the two matrices must commute. Hence $\boldsymbol{M}^{-1}\boldsymbol{Y}=\boldsymbol{Y}\boldsymbol{M}^{-1}$. Since (5.1) is true for every positive-definite, symmetric $\boldsymbol{M}$, and since the set of all positive-definite, symmetric matrices is irreducible (Fact 1), the matrix $\boldsymbol{Y}$ must be a scalar. This completes the application of the corollary to Schur's lemma as stated in the last section (Fact 2).

With the matrix $\boldsymbol{Y}$ assuming the form of a scalar matrix $\lambda(\boldsymbol{H})\boldsymbol{I}$, we can write every term in (5.1) in the form of a total differential, i.e.,

$$\operatorname{tr}\{\underline{\underline{H}}\,\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{H}}^T\underline{\underline{M}}\underline{\underline{H}})\underline{\underline{H}}^T d\underline{\underline{M}}\}=d(\hat{U}(\underline{\underline{H}}^T\underline{\underline{M}}\underline{\underline{H}})), \tag{5.2}$$

$$\operatorname{tr}\{\hat{U}_{,\underline{\underline{C}}}(\underline{\underline{M}})\quad d\underline{\underline{M}}\}=d(\hat{U}(\underline{\underline{M}})), \tag{5.3}$$

$$\operatorname{tr}\{\lambda(\underline{\underline{H}})\underline{\underline{M}}^{-1}\quad d\underline{\underline{M}}\}=d(\lambda(\underline{\underline{H}})\ln(\det \underline{\underline{M}})). \tag{5.4}$$

For an exposition of the use of matrix notation in the calculus of differentiable functions whose

arguments are square matrices, see e.g., [6]. For our purposes here, it is sufficient to list the following formulas with which (5.2), (5.3), and (5.4) are derived:

$$\text{Given } \epsilon = \hat{\epsilon}(\underline{\underline{A}}) \qquad \text{, we have } d\hat{\epsilon} = \text{tr}\,[\hat{\epsilon}_{,\underline{\underline{A}}}^{T} d\underline{\underline{A}}]. \tag{5.5}$$

$$\text{Given } \hat{\epsilon}(\underline{\underline{A}}) = \det \underline{\underline{A}} \qquad \text{, we have } \hat{\epsilon}_{,\underline{\underline{A}}} = (\det \underline{\underline{A}})(\underline{\underline{A}}^{-1})^{T}. \tag{5.6}$$

$$\text{Given } \hat{\epsilon}(\underline{\underline{A}}) = \hat{\phi}\,(\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{D}}), \text{ we have } \hat{\epsilon}_{,\underline{\underline{A}}} = \underline{\underline{B}}^{T}\hat{\phi}_{,(\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{D}})}\underline{\underline{D}}^{T}. \tag{5.7}$$

The identity (5.1) can now be integrated without difficulty:

$$\hat{U}(\underline{\underline{H}}^{T}\underline{\underline{M}}\underline{\underline{H}}) - \hat{U}(\underline{\underline{M}}) = \lambda(\underline{\underline{H}}) \ln (\det \underline{\underline{M}}) + \beta(\underline{\underline{H}}). \tag{5.8}$$

Detailed steps for the determination of the scalar functions $\lambda(\boldsymbol{H})$ and $\beta(\boldsymbol{H})$, are given in [3]. It is the main purpose of this expository article to demonstrate that without the mathematical results in the theory of irreducible sets of matrices, an identity on the scalar function $\hat{U}$ in the form (5.8) would not have been derived.

6. Significance of New Result Based on Identity (5.8)

An important contribution an applied mathematician can make in the field of science and engineering is to reduce the total number of variables in a given problem through a series of rigorous arguments, each of which can be further examined for its consistency with experiments. An example of this was given in the introduction of this paper where the strain energy function depends on six components of a symmetric matrix $\boldsymbol{U}$ instead of the nine components of the matrix $\boldsymbol{F}$. It is not surprising to many mathematicians that further simplification is possible by having the strain energy function to depend only on the three principal invariants of the symmetric matrix $\boldsymbol{U}$. The physical basis for the reduction of the number of variables from six to three is known as the condition of isotropy, where the hyperelastic material responds to an arbitrary deformation with no preference to its own orientation in an undistorted state.[3] A rigorous characterization of an isotropic, hyperelastic material requires the experimental determination of a strain energy function, say, $\boldsymbol{W} = \hat{\boldsymbol{W}}(L_1, L_2, L_3)$,where L_1, L_2, L_3 are some special combinations of the eigenvalues of the symmetric matrix $\boldsymbol{U}$. Recently Penn [7] reported the results of a series of experiments on the deformation of a peroxide vulcanized, pure-gum, natural rubber. He concluded from his experiments that the strain energy function, in general, cannot be separated as a sum of two parts:

$$\boldsymbol{W} = \hat{\boldsymbol{W}}(L_1, L_2, L_3) \neq F(L_1, L_2) + G(L_3). \tag{6.1}$$

In attempting to explain this experimental result, Fong and Simmons [3] studied the thermodynamic restrictions of a theory due to Bernstein, Kearsley, and Zapas [8, 9]. The theory was motivated by that of hyperelastic materials by replacing, among other things, the strain energy function $\hat{\boldsymbol{W}}$ with a more general, time-dependent function, $U = \hat{U}(\boldsymbol{C}(t, \tau), t-\tau)$, where t and τ denote, respectively, the present and some past time between $-\infty$ and t. An identity as given in (1.2) on the gradient of the function $\hat{U}$ was derived, and Schur's lemma was applied in arriving at the identity (5.8) on $\hat{U}$. The significance of (5.8), as discussed in [3] and [10], is best described in terms of a decomposition result based on (5.8):

$$\hat{U}(L_1, L_2, L_3, \zeta) = F(L_1, L_2, \zeta) + G(L_3, \zeta) + H(L_1, L_2, \zeta) \ln L_3. \tag{6.2}$$

[3] For a thorough treatment of the notion of isotropy, see, e.g., Truesdell & Noll [1].

Since the Bernstein-Kearsley-Zapas' theory is known to describe responses of hyperelastic materials for some special forms of $\hat{U}$, it is conceivable that Penn's data [7] can be explained with an analogous decomposition on the strain energy function $\hat{W}$:

$$W = \hat{W}(L_1, L_2, L_3) = F(L_1, L_2) + G(L_3) + H(L_1, L_2) \ln L_3. \tag{6.3}$$

Further significance of the reduction of the form of $\hat{W}$ as stated in (6.3) will appear in a forthcoming paper [10].

I thank E. A. Kearsley, M. Newman, H. J. Oser, R. W. Penn, and L. J. Zapas, for their generous help and critical comments in the preparation of this exposition.

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(Paper 77B3&4–380)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 3&4, July–December 1973

Remarks on a Problem of Rademacher in the Theory of Modular Forms*

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(May 17, 1973)

Rademacher quite some time ago posed the question of deriving the classical functional equation of $1/\eta(z)$ ($\eta(z)$ is the Dedekind modular form) directly from the exact expression he had found for the partition function, $p(n)$, which arises as the Fourier coefficient of $1/\eta(z)$. Although he had been able to solve a similar problem for the absolute invariant $J(z)$, Rademacher was unable to solve the problem for $1/\eta(z)$. We here relate this question to some more recent results of Douglas Niebur, which reduce the problem to one of identically zero Poincaré series of degree $-5/2$.

Key words: Dedekind function; modular form; partition function; Poincaré series.

1. In [6][1] Rademacher introduced a method for recapturing the functional equation

$$J(-1/z) = J(z)$$

of the well-known modular invarient, $J(z)$, directly from the explicit representation of the Fourier coefficients of $J(z)$ that he had obtained earlier [5]. Subsequently, Rademacher quite naturally raised the question of finding an analogous method for deriving the functional equation of the modular form $\eta^{-1}(z)$ from the exact formula he had found for the partition function, $p(n)$, which is generated as the Fourier coefficient of $\eta(z)^{-1}$. It is the purpose of this note to consider the relationship of this problem to the results of Niebur [3]. We shall show that, in one sense at least, Niebur's results (which are quite far-reaching and deep) solve Rademacher's problem; in another sense the problem remains open. The apparent paradox will be explained in due course.

2. The *Dedekind function* $\eta(z)$ is defined by the infinite product.

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} (1-e^{2\pi imz}), \qquad \operatorname{Im} z > 0. \tag{1}$$

A simple combinatorial argument [1, pp. 32-34] shows that

$$\eta(z)^{-1} = e^{2\pi i(23/24)z} \left\{ e^{-2\pi iz} + \sum_{n=0}^{\infty} p(n+1) e^{2\pi inz} \right\}, \tag{2}$$

where $p(n)$ is the number of unrestricted partitions of n into positive integers. Now it is obvious from either (1) or (2) that

$$\eta(z+1) = e^{\pi i/12} \eta(z), \tag{3}$$

AMS Subject Classifications: 10D05; 1001.
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[1] Figures in brackets indicate the literature references at the end of this paper.

while it is a deep property of $\eta(z)$ that

$$\eta(-1/z) = e^{-\pi i/4}\, z^{1/2}\, \eta(z), \qquad \operatorname{Im} z > 0, \tag{4}$$

where $-\pi \leq \arg z < \pi$ is the convention adopted for calculating $z^{1/2}$ [1, pp. 41–43].

It follows from (1), (3), and (4) that $\eta(z)$ is a *modular cusp form of degree* $-1/2$; that is,

$$\eta(iy) \to 0, \text{ as } y \to +\infty, \tag{5}$$

and

$$\eta(Mz) = v_\eta(M)\ (cz+d)^{1/2}\, \eta(z), \tag{6}$$

for all $Mz = (az+b)/(cz+d)$, with a, b, c, d rational integers such that $ad - bc = 1$. Here $v_\eta(M)$ is a 24th root of unity, independent of z. The collection $\{v_\eta(M)\}$ is called the *multiplier system of* $\eta(z)$. It follows that $\eta(z)^{-1}$ is a modular form of degree 1/2, with multiplier system $\bar{v}_\eta$; $\eta(z)^{-1}$ has a *pole* at $i\infty$, in the sense that

$$|\, \eta(iy)^{-1}\, | \to +\infty, \text{ as } y \to +\infty. \tag{7}$$

Using (2) and (4) and a refined version of the "circle method," Rademacher [4] derived the exact formula

$$p(n) = \frac{e^{\pi 1/4}}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left\{ \frac{\sinh(\nu\sqrt{n-1/24}\,/k)}{\sqrt{n-1/24}} \right\}, \tag{8}$$

where $\nu = \pi\ \sqrt{2/3}$ and

$$A_k(n) = \sum_{\substack{h \pmod k \\ (h,k)=1}} v_\eta(M_{k,-h}) \exp\left\{ -\frac{2\pi i}{k} \left[\left(n - \frac{1}{24}\right) h + \frac{1}{24} h' \right] \right\}; \tag{9}$$

here $M_{k,-h} = \begin{pmatrix} h' & k' \\ k & -h \end{pmatrix}$, with $-hh' - kk' = 1$.

Rademacher's problem may be stated as follows. Suppose we regard the function $\eta(z)^{-1}$ as being *defined* by (2), (8), and (9), ignoring the definition (1) and the number-theoretic interpretation of $p(n)$ as the partition function. Show *from this definition* that $\eta(z)^{-1}$ is a modular form of degree 1/2 and multiplier system $\bar{v}_\eta$, that is, that

$$\eta(Mz)^{-1} = \bar{v}_\eta(M)\,(cz+d)^{-1/2}\eta(z)^{-1}, \tag{10}$$

with M as before. Since it is obvious from (2) that

$$\eta(z+1)^{-1} = e^{-\pi i/12}\eta(z)^{-1},$$

the problem reduces to showing that

$$\eta(-1/z)^{-1} = e^{\pi i/4} z^{-1/2} \eta(z)^{-1}, \qquad \operatorname{Im} z > 0. \tag{11}$$

3. The results of Niebur are valid for all automorphic forms of positive degree on H-groups, and, in particular, apply to the function $\eta(z)^{-1}$, as defined by (2) and (8). In this case Niebur's method

yields the following functional equation directly from the series (2), (8) [3, Theorem 3.2]:

$$e^{-\pi i/4}z^{1/2}\eta(-1/z)^{-1} - \eta(z)^{-1} = \left[\alpha \int_0^{i\infty} G(\tau)(\tau - \bar{z})^{1/2} d\tau\right]^-,$$

where α is a nonzero constant, $[\beta]^-$ denotes the complex conjugate of β, the path of integration is the y axis, and $G(\tau)$ is the *Poincaré series* defined by

$$G(\tau) = \sum \frac{\exp\left\{\frac{2\pi i}{24}\frac{a\tau + b}{c\tau + d}\right\}}{v_\eta(M)(c\tau + d)^{5/2}}, \qquad \text{Im } \tau > 0. \tag{13}$$

Here the sum is taken over all pairs of relatively prime integers c, d and a, b are rational integers chosen such that $ad - bc = 1$; also, $M\tau = (a\tau + b)/(c\tau + d)$ and v_η is the multiplier system of $\eta(z)$.

Now (11) follows from (12) if it can be shown that

$$\int_0^{i\infty} G(\tau)\ (\tau - \bar{z})^{1/2} d\tau \equiv 0, \tag{14}$$

and Niebur has observed that (14) is equivalent to $G(\tau) \equiv 0$ [3, Theorem 4.1]. Thus Rademacher's problem is reduced to showing that $G(\tau) \equiv 0$. On the other hand, $G(\tau)$ is a modular cusp form of dimension $-5/2$ and multiplier system v_η (cf. the discussion in [2, pp. 272–280]), and we may show that such a modular form is necessarily $\equiv 0$. This can be done as follows. With $\eta(z)$ defined by (1), the function $f(z) = G(z)/\eta(z)$ is an *entire* modular form of degree -2 and multiplier system identically 1. That is,

$$f(Mz) = (cz + d)^2 f(z), \tag{15}$$

with M as in (6) and $\text{Im } z > 0$, and

$$|f(iy)| \text{ has a finite limit as } y \to +\infty. \tag{16}$$

It is not too difficult to show that $f(z)$ satisfying (15) and (16) is identically 0 [2, p. 216]. Thus $G \equiv 0$ and Rademacher's problem is solved.

But only in a sense, for to maintain the spirit within which Rademacher proposed the problem, one should show that $G \equiv 0$ *directly from the expression* (13). This is as yet unresolved and apparently exceedingly difficult.

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(Paper 77B3&4–381)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematical Sciences
Vol. 77B, Nos. 3 & 4, July–December 1973

The Characterizations of $(A_q(U))$*

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(May 17, 1973)

For $q>1$ Bers defines a Banach space $A_q(U)=\left\{ f\epsilon H(U) \mid \int_U |f(u)|(1-|u|^2)^{q-2}dxdy<\infty \right\}$ and shows that any bounded linear functional Λ on $A_q(U)$ may be represented as $\Lambda(f)=\int_U f(u)\bar{G}(u)(1-|u|^2)^{2q-2}dxdy$ where $G\epsilon B_q(U)=\{h\epsilon H(U) \mid \sup_{u\epsilon U} |h(u)|(1-|u|^2)^q<\infty\}$ and is unique. This work is done in [1][1]. Duren, Romberg, and Shields, pursuant to their work on H^p for $p<1$, define a Banach space $B^p(p<1)=\left\{ f\epsilon H(U) \mid \int_0^{2\pi}\int_0^1 |f(re^{i\theta})|(1-r)^{1/p-2}drd\theta<\infty \right\}$. They show that a bounded linear functional Λ on B^p may be uniquely represented as $\Lambda(f)=\lim_{r\to 1}\int_0^{2\pi} f_r(e^{i\theta})\bar{g}(e^{i\theta})d\theta$ where

(i) $f_r(e^{i\theta})=f_r(re^{i\theta})$
(ii) $g\epsilon\Lambda=\{h\epsilon H(U) \mid g$ is continuous on $\bar{U}\}$ and $g^{(n-1)}$, and $n-1$ st derivative of g, is in $\Lambda_\alpha=\{h\epsilon H(U)|h'(re^{i\theta})=0((1-r)^{\alpha-1})\}$. (Here $\alpha=1/p-n$ where $n<1/p<n+1$, so $\alpha\neq 0$. If $1/p=n+1$, the conditions on g are: $g\epsilon\Lambda$, $g^{(n-1)}\epsilon\Lambda_*=\{h\epsilon H(U)|h''(re^{i\theta})=0((1-r)^{-1})\}$.) This work appears in [2].

In this paper, after showing that $B^p\cong A_q(U)$ with $1/p=q$, we derive the relationship between G and g, namely:

$$G(z)=\sum_{k=0}^{2n+1} A_k\cdot g^{(2k+1)}(z)\cdot z^{2k+1},$$

($|z|<1$) where A_i are constants, $A_{2n+1}\neq 0$. (in this case $q=1/p$ is an integer. The Theorem is slightly different if q is not an integer.)

Key words: Automorphic functions; Hardy spaces.

We begin by showing that B^p is isomorphic to $A_q=A_q(U)$ with $1/p=q$. (the mapping is the identity.)

PROOF: Take $f\epsilon B^p$.

$$\int_U |f|(1-|u|^2)^{q-2}dxdy=\frac{1}{2\pi}\int_0^1 |f|(1-r^2)^{q-2}rdrd\theta\leq\frac{\max(1,2^{q-2})}{2\pi}\int_0^{2\pi}\int_0^1 |f|(1-r)^{q-2}drd\theta.$$

This shows that $k\|f\|_{B^p}\geq\|f\|_{A_q}$. So $B^p\subset A_q$, with a continuous injection. Now it is easy to verify that if $f\epsilon A_q$ then $f\epsilon B^p$. Thus we have $i: B^p\to A_q$, i the injection, is continuous and onto. The Open Mapping Theorem implies that i^{-1} is continuous, completing the proof that $B^p\cong A_q$, with $1/p=q$.

We turn to the Duren, Romberg and Shields representation. We hereafter will only refer to A_q, dropping all reference to B^p. If $f(z)=z^n$, $n\geq 0$ (which is certainly in A_q).

$$\Lambda(z^n)=\lim_{r\to 1}\int_0^{2\pi} r^n e^{in\theta}\bar{g}(e^{i\theta})d\theta=\int_0^{2\pi} e^{in\theta}\bar{g}(e^{i\theta})d\theta,$$

AMS Subject Classification: 30A58; 30A78.
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since $g \in A$.

But now

$$\int_0^{2\pi} e^{in\theta}\bar{g}(e^{i\theta}) = \lim_{r\to 1}\int_0^{2\pi} e^{in\theta}\bar{g}_r(e^{i\theta})d\theta,$$

by the Lebesgue Dominated Convergence Theorem since $\|g_r\|_\infty \leq \|g\|_\infty$. Now, letting $g(z) = \sum_{k=0}^{\infty} b_k z^k$,

$$\int_0^{2\pi} e^{in\theta}\bar{g}_r(e^{in\theta})d\theta = \int_0^{2\pi} e^{in\theta}\sum_0^\infty \bar{b}_k r^k e^{-ik\theta}d\theta$$

$$= \sum_0^\infty \int_0^{2\pi} (\bar{b}_k r^k e^{in\theta - ik\theta})d\theta$$

(since the power series for g_r converges absolutely in $\bar{U}$)

$$= 2\pi\bar{b}_n r^n$$

and

$$\lim_{r\to 1}\int_0^{2\pi} e^{in\theta}\bar{g}_r(e^{i\theta})d\theta = 2\pi\bar{b}_n.$$

We have proved

LEMMA 1: *If* $g(z) = \sum_{k=0}^{\infty} b_k z^k$,

$$\Lambda(z^n) = 2\pi\bar{b}_n.$$

Now we turn to the Bers representation of $\Lambda(z^n)$. *If*

$$G(z) = \sum a_k z^k,\ \lambda(z) = (1-|z|^2)$$

$$\Lambda(z^n) = \int\int_U z^n\overline{G}(z)\lambda^{2q-2}dxdy$$

$$= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} r^n e^{in\theta}\sum_0^\infty \bar{a}_k r^k e^{-ik\theta}(1-r^2)^{2q-2}rdrd\theta.$$

Define $G_\rho(z) = G(\rho z)$. Then

$$\sup_{|z|=s}|G_\rho(z)\lambda^q(z)| = \sup_{|z|=s}|G_\rho(z)|\lambda^q(s)$$

(since λ depends only on $|z|$)

$$\leq \sup_{|z|=s}|G(z)|\lambda^q(s)$$

$$= \sup_{|z|=s}|G(z)|\lambda^q(z).$$

We have shown that

$$|G_\rho(z)\lambda^q(z)| \leq \sup_{|\zeta|\leq|z|} |G_\rho(\zeta)\lambda^q(\zeta)|$$

$$\leq \sup_{|\zeta|\leq|z|} |G(\zeta)\lambda^q(\zeta)| \leq K;$$

since $G\epsilon B_q$. Then,

(*) $$\lim_{\rho\to 1}\int\int f\bar{G}_\rho\lambda^{2q-2}dxdy=\int\int f\bar{G}\lambda^{2q-2}dxdy$$

since

$$|fG_\rho\lambda^{2q-2}|=|f(G_\rho\lambda^q)\lambda^{q-2}|\leq K|f\lambda^{q-2}|,$$

by above. But since $f\epsilon A_q$, $K|f\lambda^{q-2}|$ is integrable so (*) holds. Now, as before, we compute $\int\int z^n\bar{G}_\rho\lambda^{2q-2}dxdy, n\geq 0.$

$$\int\int z^n\bar{G}_\rho\lambda^{2q-2}dxdy$$

$$=\frac{1}{2\pi}\int_0^1\int_0^{2\pi} r^n e^{in\theta}\left(\sum_0^\infty \bar{a}_k\rho^k r^k e^{-ik\theta}\right)(1-r^2)^{2q-2}rdrd\theta$$

$$=\frac{1}{2\pi}\int_0^1 (1-r^2)^{2q-2}\cdot r^{n+1}\cdot\int_0^{2\pi} e^{in\theta}\left(\sum_0^\infty a_k\rho^k r^k e^{-ik\theta}\right)d\theta dr$$

$$=\frac{1}{2\pi}\int_0^1 (1-r^2)^{2q-2}r^{n+1}\sum_0^\infty\int_0^{2\pi}\bar{a}_k\rho^k r^k e^{in\theta-ik\theta}d\theta dr$$

(since the series for G_ρ converges absolutely for $|z|=r\leq 1$)

$$=\frac{1}{2\pi}\int_0^1 (1-r^2)^{2q-2}r^{n+1}\bar{a}_n\rho^n r^n\cdot 2\pi dr$$

$$=\frac{1}{2\pi}\cdot 2\pi\bar{a}_n\rho^n\int_0^1 (1-r^2)^{2q-2}r^{2n+1}dr$$

and taking limit as $\rho\to 1$ we get

LEMMA 2: *If* $G(z)=\sum_0^\infty a_k z^k$

$$\Lambda(z^n)=\bar{a}_n\cdot\int_0^1 (1-r^2)^{2q-2}r^{2n+1}dr=\bar{a}_n\cdot c_n$$

where

$$c_n=\int_0^1 (1-r^2)^{2q-2}r^{2n+1}dr.$$

COROLLARY 3: $n \geq 0$

$$2\pi b_n = a_n c_n.$$

PROOF: By taking conjugates, and noting that c_n and 2π are real.

We shall need the following computation.

LEMMA 4: *Let*

$$p^c n = \int_0^1 r^{2n+1}(1-r^2)^p dr,$$

then

$$p^c n = \frac{n!\Gamma(p+1)}{\Gamma(p+n+2)\cdot 2}.$$

(Here $p \neq -1$.)

PROOF: By integration by parts

$$p^c n = \int_0^1 r^{2n+1}(1-r^2)^p dr$$

$$= \frac{1}{p+1}\int r^{2n-1}(1-r^2)^{p+1} dr$$

$$= \frac{n}{p+1}(p+1)^c (n-1).$$

Also, it is easy to compute

$$p^c 0 = \frac{1}{2(p+1)}.$$

Using our recursion equation and initial condition, we obtain:

(**) $$p^c n = \frac{n!}{2(p+1) \ldots (p+n+1)}.$$

Now

$$p\Gamma(p) = \Gamma(p+1).$$

So

$$[2(p+1) \ldots (p+n+1)]\Gamma(p+1) = 2\Gamma(p+n+2)$$

or

$$[2(p+1) \ldots (p+n+1)] = \frac{2\Gamma(p+n+2)}{\Gamma(p+1)}.$$

Substituting in (**), we obtain

$$p^c n = \frac{n!\Gamma(p+1)}{2\Gamma(p+n+2)}.$$

We can now prove the basic theorems. First, the case when $q = n+1$, an integer.

THEOREM 5: *Let* $q=n+1$, *then*

(1) $G(z)=A_{2n+1}g^{(2n+1)}(z)\cdot z^{2n+1}+A_{2n}g^{(2n)}(z)\cdot z^{2n}+\ldots+A_0g(z)(|z|<1)$ *where* A_i *are constants,* $A_{2n+1}\neq 0$.

PROOF: $a_m=2\pi b_m/c_m$ where

$$c_m=(2q-2)^{\overline{c}}\,m=\frac{m!\Gamma(2q-1)}{2\Gamma(2q+m)}$$

$$2q-2=2(n+1)-2=2n$$

$$c_m=\frac{m!\,2n!}{(2n+m+1)!\cdot 2}.$$

Then

$$a_m=2\pi b_m/c_m=\frac{2\pi b_m\cdot 2\cdot(2n+m+1)!}{m!\,2n!}$$

Now, $\dfrac{(2n+m+1)!}{m!}$ is a polynomial of proper degree $2n+1$. Such polynomial is a linear combination of the following $2n+2$ polynomials:

$$C_0(m)=1$$

$$C_1(m)=m$$

$$C_2(m)=m(m-1)$$

$$C_3(m)=m(m-1)(m-2)$$

$$\vdots$$

$$C_{2n+1}(m)=m(m-1)\ldots(m-2n)$$

Thus we have

$$a_m=\frac{4\pi b_m}{2n!}\cdot\sum_{i=0}^{2n+1}B_i\cdot C_i(m).$$

(B_i constant, $B_{2n+1}\neq 0$); then

$$a_mr^me^{im\theta}=\frac{4\pi}{2n!}\sum_{i=0}^{2n+1}B_iC_i(m)b_mr^me^{im\theta}$$

and, if $r<1$

$$\sum_{m=0}^{\infty}a_mr^me^{im\theta}=\sum_{m=0}^{\infty}\frac{4\pi}{2n!}\cdot\sum_{i=0}^{2n+1}B_iC_i(m)\cdot b_mr^me^{im\theta}$$

$$=\frac{4\pi}{2n!}\sum_m\sum_i B_i\cdot C_i(m)b_mr^me^{im\theta}$$

$$=\frac{4\pi}{2n!}\sum_i\sum_m B_iC_i(m)b_mr^me^{im\theta}$$

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(since $r<1$)

$$=\frac{4\pi}{2n!}\sum_i B_i \sum_{m=0}^{\infty} C_i(m) b_m r^m e^{im\theta}.$$

Now,

$$g^{(k)}(z)=\sum_{m=0}^{\infty} C_k(m) b_m r^{m-k} e^{i(m-k)\theta}$$

or

$$g^{(k)}(z)=z^{-k}\cdot\sum_m C_k(m) b_m r^m \cdot e^{im\theta}$$

or

$$z^k g^{(k)}(z)=\sum_m C_k(m) b_m r^m e^{im\theta}$$

Hence

$$G(z)=\sum_{m=0}^{\infty} a_m r^m e^{im\theta}$$

$$=\frac{4\pi}{2n!}\sum_{i=0}^{2n+1} B_i \cdot g^{(i)}(z)\cdot z^i.$$

Now, with $A_i=\dfrac{4\pi B_i}{2n!}$ we have $(1)\cdot A_{2n+1}\neq 0$ because $B_{2n+1}\neq 0$.

We now turn to the case when q is *not* an integer.

COROLLARY 6: *Let* $n<q<n+1$. *Then* $G\in B_q$ *may be written*

$$G(z)=\sum_{i=0}^{2n+1} A_i\cdot z^i\cdot g^{(i)}(z),\ A_{2n+1}\neq 0$$

where not only is $g^{(n-1)}\in\Lambda_*$, $g\in A$ *but* $g^{(2n+1)}=0\left(\dfrac{1}{(1-r)^q}\right)$.

(*We shall see shortly that* $g^{(n-1)}\in\Lambda_* \Rightarrow g^{(2n+1)}=0(1/(1-r)^{n+1}$.)

Note that we do *not* claim that this g is the same as the g associated with Λ.

PROOF: Choose β such that $q+\beta=n+1$. $G\in B_q$. $G\in B_{q+\beta}=B_{n+1}$ since $\lambda(z)\leq 1$, and $\beta>0$. Apply Theorem 5, getting $G(z)=\sum_{i=0}^{2n+1} A_i z^i g^{(i)}(z)$. The proof that $g^{(2n+1)}=0(1/(1-r)^q)$ depends on

LEMMA 7: *Say* $f\in H(U)$,

$$|f(re^{i\theta})|<k/(1-r)^l.$$

Then

$$|f'(re^{i\theta})|<k'/(1-r)^{l+1}.$$

PROOF: Pick z, $r=|z|$, $r<p<1$, $C_\rho=\{z\,|\,|z|=\rho\}$. Then

$$f'(z)=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)}{|\zeta-z|^2}\,d\zeta$$

$$|f'(z)|\leq\frac{k}{(1-\rho)^l}\cdot\frac{1}{2\pi}\cdot\int_{C_\rho}\frac{|d\zeta|}{|\zeta-z|^2}.$$

We must estimate the last integral

$$\int_{C_\rho} \frac{|d\zeta|}{|\zeta - z|^2} = \int_0^{2\pi} \frac{\rho d\theta}{|\rho e^{i\theta} - z|^2}$$

(where $\zeta = \rho e^{i\theta}$, $d\zeta = \rho i e^{i\theta} d\theta$, so $|d\zeta| = \rho d\theta$).

Now, let $\rho = (1+r)/2$. If we set $z^* = z/\rho$, $|z^*| = 2r/(1+r) < 1$, when $r < 1$. Then.

$$\int_0^{2\pi} \frac{\rho d\theta}{|\rho e^{i\theta} - z|^2} = \frac{1}{\rho} \int_0^{2\pi} \frac{d\theta}{\left| e^{i\theta} - \frac{z}{\rho} \right|^2}$$

$$= \frac{2}{1+r} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - z^*|^2}.$$

Let us look at the last integral.

$$2\pi = \int_0^{2\pi} \operatorname{Re} \frac{e^{it} + z^*}{e^{it} - z^*} dt \qquad \text{(Poisson Kernel)}$$

$$= \int_0^{2\pi} \operatorname{Re} \frac{(e^{it} + z^*)(\overline{e^{it} - z^*})}{|e^{it} - z^*|^2} dt$$

$$= \int_0^{2\pi} \operatorname{Re} \frac{1 - |z^*|^2 + (\overline{e^{it}} z^* - e^{it} \bar{z}^*)}{|e^{it} - z^*|^2} dt$$

Now $\overline{e^{it}} z^* - e^{it} \bar{z}^*$ is imaginary so

$$= \int_0^{2\pi} \frac{1 - |z^*|^2}{|e^{it} - z^*|^2} dt$$

$$= (1 - |z^*|^2) \int_0^{2\pi} \frac{dt}{|e^{it} - z^*|^2}$$

Thus

$$\frac{2}{1+r} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - z^*|^2} = \frac{2}{1+r} \cdot \frac{2\pi}{1 - |z^*|^2}$$

Now since $G \epsilon B_q, n < q < n+1$ we see that

(***) $$(1-r)^q \cdot z^i \cdot g^{(i)}(z) = 0(1)$$

for $i > 2n+1$.

$$G(z)\,(1-r)^q = 0(1) + (1-r)^q g^{2n+1}(z) \cdot z^i$$

by (***). Thus $g^{2n+1} = 0(1/(1-r)^q)$.

This concludes the proof of *Corollary 6.*

We ask the following question: Is every sum of the form

$\Sigma A_i \cdot z^i \cdot g^{(i)}(z)$ in B_q ? First, if q *is an integer*, the answer is yes. For then $g^{n-1} \epsilon \Lambda_*$ which

implies

$$\left.\begin{aligned} g^{n+1} &= 0(1/(1-r) \\ g^{n+2} &= 0(1/(1-r)^2) \\ &\vdots \\ g^{2n+1} &= 0\left(\frac{1}{(1-r)^{n+1}}\right) = 0\left(\frac{1}{(1-r)^q}\right) \end{aligned}\right\} \quad \text{successively, by Lemma 7.}$$

Thus clearly

$$\sum_{i=0}^{2n+1} A_i \cdot z^i \cdot g^{(i)}(z) \in B_q.$$

If q is *not an integer*, then not every such sum is in B_q. Only those with $g^{2n+1} = 0(1/(1-r)^q)$ will be in B_q. $g \in \Lambda_*$ only implies that $g^{2n+1} = 0(1/(1-r)^{n+1})$, and $0(1/(1-r)^{n+1})$ is a weaker condition than $0(1/(1-r)^q)$, since $q < n+1$.

The author wishes to thank Jay Stepelman without whom this work could not have begun.

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(Paper 77B3&4–382)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematical Sciences
Vol. 77B, Nos. 3 & 4, July–December 1973

Character Induced Subgroups*

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(May 17, 1973)

Given a finite group G with an irreducible character χ, define $G_\chi = \{g \in G : |\chi(g)| = \chi(1)\}$. Then $\chi(1)^2 \leq [G: G_\chi]$. We investigate the case of equality. There are applications to symmetry classes of tensors and generalized matrix functions.

Key words: Character of central type; central idempotent; Frobenius Reciprocity; group algebra; orthogonality relations; tensor product.

1. Character Induced Subgroups

Let G be a finite group with an irreducible (complex) representation $\{A(g) : g \in G\}$ and corresponding character χ. Let

$$G_\chi = \{g \in G : |\chi(g)| = \chi(1)\}.$$

One easily sees that G_χ is the normal subgroup of G consisting of those elements represented by scalars in $\{A(g)\}$. Moreover, $\lambda = \chi/\chi(1)$ is a linear character on G_χ which is invariant under conjugation by elements of G. we call G_χ the subgroup *induced* by χ.

THEOREM 1: *We have* $\chi(1)^2 \leq [G : G_\chi]$, *with equality if and only if* χ *is the only irreducible character of* G *whose restriction to* G_χ *contains* λ *as a component.*

PROOF: Let λ^G be the character of G induced by λ. Then, by the Frobenius Reciprocity Theorem, χ occurs in λ^G exactly $\chi(1)$ times. Moreover, if η is an irreducible character on G whose restriction to G_χ contains λ, then $\eta \in \lambda^G$. But, the degree of λ^G is $[G : G_\chi]$.

We might point out that if η is an irreducible character on G such that $\lambda \in \eta \mid G_\chi$, then $\eta \mid G_\chi = \eta(1)\lambda$ [8, p. 53]. In particular, $G_\chi \subset G_\eta$.

COROLLARY 1: *Let* $g \in G$ *be arbitrary. Then* λ *can be extended to a character of* $\langle G_\chi, g\rangle$, *the group generated by* G_χ *and* g. *If* $\chi(1)^2 = [G : G_\chi > 1$, *then* λ *cannot be extended to* G.

PROOF: The first statement follows because G_χ is normal and λ is invariant. The second follows from Theorem 1.

We now give another proof of Theorem 1 which leads to an apparently different case of equality. First, define the *support of* χ to be $\operatorname{supp}\chi = \{g \in G : \chi(g) \neq 0\}$.

THEOREM 1′: *We have* $\chi(1)^2 \leq [G : G_\chi]$, *with equality if and only if supp* $\chi = G_\chi$.

PROOF: Let $G = \bigcup_{i=1}^{m} g_i G_\chi$ be the coset decomposition of G with respect to G_χ. Then

$$o(G) = \sum_{g \in G} |\chi(g)|^2$$

AMS Subject Classification: 20C15
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$$= \sum_{i=1}^{m} \sum_{h \in G_\chi} |\chi(g_i)|^2$$

$$= o(G_\chi) \sum_{i=1}^{m} |\chi(g_i)|^2.$$

The case of equality in Theorem 1′ recalls the following result of Brunside

THEOREM 2: ([3, p. 322], [8, p. 93], [11, p. 41]). *Suppose that* $(o(C), \chi(1)) = 1$ *for some conjugacy class* C *of* G. *Then* $\chi(C) = 0$ or $|\chi(C)| = \chi(1)$.

THEOREM 3: *If* $A_1, \ldots, A_r$ *are the inequivalent irreducible representations of* G *and if* G_i *is the subgroup induced by the character of* A_i, *then*

$$\bigcap_{i=1}^{r} G_i = Z(G), \tag{1}$$

the center of G. *If* A_i *is faithful, then* $G_i = Z(G)$.

PROOF: The group G_i is the inverse image of the center of $\{A_i(g)\}$. Equation (1) follows from an examination of the center of the regular representation.

THEOREM 4: *Suppose* $\chi(1)^2 = [G : G_\chi]$. *Let* $K = \{g \in G : \chi(g) = \chi(1)\}$. *Then* $(G/K)_\chi \cong G_\chi/K$ *is cyclic.*

PROOF: If $G = \bigcup_{i=1}^{m} g_i K$ is a coset decomposition of G, $g_i K \to A(g_i)$ defines an irreducible representation of G/K. Thus, χ is a character of G/K and $(G/K)_\chi \cong G_\chi/K$. But, G_χ/K is isomorphic to the group of values of λ, a subgroup of the $o(G_\chi)$th roots of unity.

The group G is of *central type* if $\chi(1)^2 = [G : Z(G)]$ for some irreducible character χ [6], [7]. Since $Z(G) \subset G_\chi$, if G is of central type with respect to χ, then $\chi(1)^2 = [G : G_\chi]$.

THEOREM 5: *If* $\chi(1)^2 = [G : G_\chi]$, *then there exist irreducible characters* $\chi_p, \chi_q, \ldots, \chi_r$ *on the Sylow subgroups* $S_p, S_q, \ldots, S_r$ *of* G, *respectively, such that* $(S_t)_{\chi_t} = S_t \cap G_\chi$, $t = p, q, \ldots, r$, *and* $[S_t : (S_t)_{\chi_t}] = \chi_t(1)^2$, $t = p, q, \ldots, r$.

The following simple proof of Theorem 5 was communicated by Professor DeMeyer to the first author [5]: If $\chi(1)^2 = [G : G_\chi]$, then $G/\ker \chi$ is a group of central type with character χ. Thus, Theorem 5 follows from [7, Theorem 2].

2. Examples

(a) If χ is linear, then $G_\chi = G$ and $\chi(1)^2 = [G : G_\chi]$.

(b) Let G be a finite group with normal subgroup H. Let $\{B(gH) : g \in G\}$ be any irreducible representation of G/H. Then $\{B(gH)\}$ gives rise to an irreducible representation of G, $A(g) = B(gH)$. Let χ be the character afforded by $\{A(g)\}$. Since H is in the kernel of $\{A(g)\}$, $H \subset G_\chi$.

(c) Let G be a finite group with irreducible character χ. Let L be any group with linear character η. Then $H = G \times L$ is endowed with the irreducible character ζ, $\zeta(g, 1) = \chi(g)\,\eta(1)$. Moreover, $H_\zeta = G_\chi \times L$, and $[H : H_\zeta] = [G \times L : G_\chi \times L] = [G : G_\chi]$.
If $[G : G_\chi] = \chi(1)^2$ then $[H : H_\zeta] = \zeta(1)^2$. If L is not abelian, then $Z(H) \subsetneq H_\zeta$.

(d) Let p be a prime. Suppose G is the group of order p^3 generated by elements g and h with defining relations

$$g^{p^2} = h^p = 1, h^{-1}gh = g^{p+1}.$$

Let A be the representation of G of degree p given by

$$A(g) = \omega^{1/p} \operatorname{diag}(1, \omega, \ldots, \omega^{p-1}),$$

$$A(h) = \begin{bmatrix} 0 & 0 & . & . & . & 0 & 1 \\ 1 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & . & 0 & 0 \\ & & & . & . & . & \\ 0 & 0 & . & . & . & 1 & 0 \end{bmatrix}$$

where $\omega = \exp(2\pi i / p)$. Let χ be the character afforded by A. Then, $G_\chi = Z(G) = \langle g^p \rangle$ is of order p, and $\chi(g^p) = p\omega$. (For details, see [11, p. 43].)

3. Central Idempotents

The authors believe the material of this section to be of independent interest, but it may also be viewed as lemmata for Theorem 8 in the next section.

Let KG denote the complex group algebra of G. Define

$$t(G, \chi) = \frac{\chi(1)}{o(G)} \sum_{g \in G} \chi(g) g.$$

Suppose $\chi = \chi_1, \ldots, \chi_r$ are inequivalent, irreducible characters on G. Then $t(G, \chi_1), \ldots, t(G, \chi_r)$ are annihilating idempotents and

$$t(G, \chi_1) + \ldots + t(G, \chi_r) = 1 \epsilon G.$$

Moreover, $t(G, \chi_1), \ldots, t(G, \chi_r)$ span $Z(KG)$, the center of KG. Finally, $t(G, \chi_i)$ generates the simple two sided ideal in KG to which χ_i corresponds [4, pp. 233–236], [1, p. 83].

THEOREM 6: *We have* $t(G_\chi, \lambda) = \Sigma t(G, \eta)$, *where the summation is over those inequivalent, irreducible characters* η *of* G *whose restriction to* G_χ *contains* λ *as a component. In particular,* $t(G_\chi, \lambda) = t(G, \chi)$ *if and only if* $[G : G_\chi] = \chi(1)^2$.

PROOF: We claim first, that $t(G_\psi, \lambda) \epsilon Z(KG)$. Take $g \epsilon G$. Then

$$\begin{aligned} g^{-1} t(G_\chi, \lambda) g &= \frac{1}{o(G_\chi)} \sum_{h \epsilon G_\chi} \lambda(h) g^{-1} h g \\ &= \frac{1}{o(G_\chi)} \sum_{h \epsilon G_\chi} \lambda(g h g^{-1}) h \\ &= t(G_\chi, \lambda), \end{aligned}$$

because G_χ is normal in G and λ is invariant under conjugation by elements of G.

Thus, there exist complex numbers $a_1, \ldots, a_r$ such that

$$t(G_\chi, \lambda) = \sum_{i=1}^{r} a_i t(G, \chi_i). \tag{2}$$

Considered as linear operators on KG (multiply on the left), $t(G, \chi_i)$, and $t(G_\chi, \lambda)$ are hermitian with respect to the inner product which makes $G \subset KG$ an *o. n.* basis. Since they are also idempotent, we may view them as orthogonal projections on KG. Since $t(G, \chi_1), \ldots, t(G, \chi_r)$ are mutually annihilating and belong to $Z(KG)$, all the a_i's must be 1 or 0. Indeed,

$$a_i = \begin{cases} 1, \text{ if } t(G_\chi, \lambda)t(G, \chi_i) = t(G, \chi_i), \\ 0 \text{ otherwise.} \end{cases}$$

Thus, $a_i = 1$ if and only if $t(G_\chi, \lambda)t(G, \chi_i)$ is not zero. But, if ζ is the character of the regular representation of G,

$$\begin{aligned} &\text{rank } (t(G_\chi, \lambda)t(G, \chi_i)) \\ &= \text{trace } (t(G_\chi, \lambda)t(G, \chi_i)) \\ &= \frac{\chi_i(1)}{o(G_\chi)o(G)} \sum_{h \in G_\chi} \lambda(h) \sum_{g \in G} \chi_i(h^{-1}g)\zeta(g) \\ &= \frac{\chi_i(1)}{o(G_\chi)} \sum_{h \in G_\chi} \lambda(h)\chi_i(h^{-1}). \end{aligned}$$

This last expression is $\chi_i(1)$ times the number of occurrences of λ in $\chi_i | G_\chi$.

4. Symmetry Classes of Tensors

Let V be an n-dimensional complex inner product space. Take $m \leq n$. Denote by $\otimes^m V$ the mth tensor power of V. If $v_1, \ldots, v_m \in V$, then $v_1 \otimes \ldots \otimes v_m \in \otimes^m V$ will denote their tensor product. If $(.,.)$ denotes the inner product on V, then

$$(v_1 \otimes \ldots \otimes v_m, v_1' \otimes \ldots \otimes v_m') = \prod_{t=1}^{m} (v_t, v_t') \tag{3}$$

is an inner product on $\otimes^m V$.

Let S_m denote the symmetric group of degree m. Given $g \in S_m$, there is a linear operator, $P(g)$ on $\otimes^m V$ such that

$$P(g^{-1})v_1 \otimes \ldots \otimes v_m = v_{g(1)} \otimes \ldots \otimes v_{g(m)},$$

for all $v_1, \ldots, v_m \in V$. Observe that $g \to P(g)$ is a representation of S_m.

Suppose, now, that G is a subgroup of S_m and χ is an irreducible character on G. Let

$$T(G, \chi) = \frac{\chi(1)}{o(G)} \sum_{g \in G} \chi(g)P(g).$$

With respect to the inner product, (3), $T(G, \chi)$ is hermitian. It is a consequence of the orthogonality relations for characters that $T(G, \chi)$ is idempotent. Thus, $T(G, \chi)$ is an orthogonal projection. Let $V_\chi^m(G)$ denote the range of $T(G, \chi)$. Then $V_\chi^m(G)$ is called a *symmetry class of tensors.*

Let V^m denote the mth cartesian power of V. Define

$$f\colon V^m \to V_\chi^m(G) \text{ by } f(v_1, \ldots, v_m) = T(G, \chi)v_1 \otimes \ldots \otimes v_m.$$

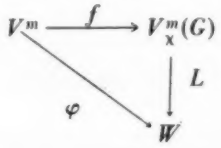
clearly, f is m-linear. If χ is a linear character, then f is symmetric with respect to G and χ, i.e., if $g \in G$, then

$$f(v_{g(1)}, \ldots, v_{g(m)}) = \chi(g) f(v_1, \ldots, v_m). \tag{4}$$

This follows from the observation that

$$T(G, \chi)P(g^{-1}) = \chi(g)T(G, \chi). \tag{5}$$

From (4), and the Universal Factorization Property for tensor spaces, it follows that: If W is any complex vector space and φ: $V^m \to W$ is any m-linear function, symmetric with respect to G and χ, then there exists a unique linear L: $V_\chi^m(G) \to W$ such that $\varphi = L \cdot F$, i.e., such that the diagram



is commutative.

For characters of degree greater than one, (5) does not hold. We are motivated to define

$$G' = \{g \in S_m : T(G, \chi)P(g^{-1}) \in \langle T(G, \chi) \rangle\},$$

i.e., G' is the set of elements $g \in S_m$ such that $T(G, \chi)P(g^{-1})$ is a multiple of $T(G, \chi)$. For $g \in G'$, write

$$T(G, \chi)P(g^{-1}) = c(g)T(G, \chi).$$

THEOREM 7: *We have* $G' = G_\chi$ *and* $c = \lambda$.

PROOF: Suppose, first, that $g \in G_\chi$. Then

$$\begin{aligned} T(G, \chi)P(g^{-1}) &= \frac{\chi(1)}{o(G)} \sum_{h \in G} \chi(h)P(h)P(g^{-1}) \\ &= \frac{\chi(1)}{o(G)} \sum_{h \in G} \chi(hg)P(h) \\ &= \lambda(g)T(G, \chi). \end{aligned}$$

Thus, $G_\chi \subset G'$ and $c|G_\chi = \lambda$.

Next, we prove that G' is a group and c is a linear character on it.

Since $P(1)$ is the identity, $1 \in G'$ and $c(1) = 1$. Since $P(g^{-1}) = P(g)^{-1}$, $g \in G'$ implies $g^{-1} \in G'$ and $c(g^{-1}) = c(g)^{-1}$. Finally, if $g, h \in G'$, then

$$\begin{aligned} T(G, \chi)P(h^{-1}g^{-1}) &= c(h)T(G, \chi)P(g^{-1}) \\ &= c(h)c(g)T(G, \chi). \end{aligned}$$

It follows that $gh \in G'$ and $c(gh) = c(g)c(h)$.

Now let $g \in G'$. Let $v_1, \ldots, v_m$ be an *o.n.* set in V. We compute the same inner product in two ways to obtain

$$(f(v_{g(1)}, \ldots, v_{g(m)}), c(g)f(v_1, \ldots, v_m))$$

$$= \overline{c(g)}\,\frac{\chi(1)}{o(G)} \sum_{h \in G} \chi(h) \prod_{t=1}^{m} (v_{g(t)}, v_{h(t)})$$

$$= \begin{cases} \overline{c(g)}\,\dfrac{\chi(1)}{o(G)}\,\chi(g), & g \in G \\ 0, & g \notin G, \end{cases}$$

and $||c(g) f(v_1, \ldots, v_m)||^2 = \chi(1)^2/o(G)$. Thus,

$$G' \subset \{g \in G\colon |\chi(g)| = \chi(1)\} = G_\chi.$$

Our next result involves the generalized matrix functions of Schur [12]. If $A = (a_{ij})$ is an m-square complex matrix, define

$$d_G^\chi(A) = \sum_{g \in G} \chi(g) \prod_{t=1}^{m} a_{tg(t)}.$$

THEOREM 8: *Let* $A = (a_{ij})$ *be an* m*-square matrix. Then*

$$\frac{1}{o(G_\chi)}\, d_{G_\chi}^{\lambda}(A) = \sum \frac{\eta(1)}{o(G)}\, d_G^{\eta}(A),$$

where the summation is over those inequivalent, irreducible characters η *of* G *whose restriction to* G_χ *contains* λ *as a component.*

When $G_\chi = \{1\}$, this result collapses to an identity of Freese [9, eq (8)].

COROLLARY 2: *Suppose* A *is positive semidefinite hermitian. Then*

$$\frac{1}{o(G_\chi)}\, d_{G_\chi}^{\lambda}(A) \geq \frac{\chi(1)}{o(G)}\, d_G^{\chi}(A).$$

If A *is positive definite, equality obtains if and only if* $[G\colon G_\chi] = \chi(1)^2$.

PROOF: Schur proved [11] that $d_G^\eta(A) \geq 0$, with strict inequality if A is positive definite. Thus, the result is an immediate consequence of the theorem.

The inequality in Corollary 2 is of a type recently studied by Botta [2] and others.

PROOF of THEOREM 8: Extend $\sigma \to P(\sigma)$ linearly to a representation $\bar{P}$ of KG. Then $\bar{P}$ is a homomorphism of the algebra KG onto the operator algebra generated by $\{P(g)\colon g \in G\}$.

(Indeed, since $m \leq n$, $V_\eta^m(G) \neq 0$ for every irreducible character η of G [10, Lemma 1]. But, the dimension of $V_\eta^m(G)$ is trace $T(G, \eta)$, which is the number of occurrences of η in the character of the representation $g \to P(g)$. Thus, every irreducible character of G is a component of the character $\{\operatorname{tr} P(g)\}$. It follows that $\bar{P}$ is, in fact, an isomorphism [1, p. 69].)

Since $T(G, \chi_i) = \bar{P}(t(G, \chi_i))$, we may apply Theorem 6 to obtain

$$T(G_\chi, \lambda) = \sum^{*} T(G, \eta) \tag{6}$$

where the summation is as it was there.

Let, now, $v_1, \ldots, v_m, v_1', \ldots, v_m'$ be vectors such that $a_{ij} = (v_i, v_j')$. Observe that

$$(T(G, \eta)\, v_1 \otimes \ldots \otimes v_m, v_1' \otimes \ldots \otimes v_m') =$$
$$= \frac{\eta(1)}{o(G)} \sum_{g \in G} \eta(g) \prod_{t=1}^{m} (v_t, v_{g(t)}')$$

$$= \frac{\eta(1)}{o(G)} d_G^{\eta}(A).$$

The result follows from (6).

We might remark that eq (6) also shows that

$$V_{\lambda}^{m}(G_X) = \perp V_{\eta}^{m}(G),$$

the orthogonal direct sum of those symmetry classes corresponding to the irreducible characters η of G whose restriction to G_X contains λ.

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(Paper 77B3&4–383)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 3 & 4, July-December 1973

Performance Testing of a FORTRAN Library of Mathematical Function Routines—A Case Study in the Application of Testing Techniques

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This paper describes an application of testing methodology and techniques developed by the authors to aid in improving the quality of mathematical software. These techniques differ radically in several aspects from techniques previously used, the most important difference being that the testing is not based exclusively on random arguments. Instead, throughout the range of each function use is made of special arguments that are designed to detect programming errors and to test the performance of an algorithm in different regions. The function values are tested against reference values which are stored on reference tapes generated by a highly authenticated system of subroutines.

Since the effectiveness of such a testing system in discovering errors and performance limitations can be fully ascertained only through actual use, we report the results of employing our system to test an existing FORTRAN library of mathematical function routines. Specific aspects of the numerical accuracy of the library used in this test case are discussed in order to illustrate the effectiveness of a well-designed testing system as an analytic tool for the evaluation of mathematical software.

Since documentation provides information necessary to perform testing and contains specifications that reflect the results of testing, our study includes comments on the documentation.

No information on timing or storage requirements is presented in this case study.

Key words: Automated testing; bit comparison; FORTRAN library; function validation; mathematical functions; performance tests.

1. Introduction

This paper presents the results of a case study of a FORTRAN library of mathematical function routines which serves to demonstrate the effectiveness of a testing program developed by the authors [1][1]. To achieve this purpose we include in the paper a general evaluation of the library and its documentation as well as data on its numerical accuracy. Programming errors discovered by the application of our techniques are pointed out and suggestions for improvement are made. The particular library which served as our test case is the UNIVAC 1108 FORTRAN library of mathematical functions. This library was released by UNIVAC for the EXEC 8 operating system in 1971; the RL 24 version reported on in this paper is a release that contains modifications enabling it to run under EXEC II.

This case study should not be construed as a validation or certification of the manufacturer's software and the authors would like to stress the fact that no endorsement of the manufacturer's library is implied.

AMS Subject Classification: 68A10.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Testing Techniques

Below is a short discussion of some of the salient features of our testing approach.

1. The library function values were tested by comparing them with reference values. The latter were obtained through the use of a software system containing an arbitrary precision arithmetic package and a set of arbitrary precision algorithms developed by us at the National Bureau of Standards [2]. This eliminates the necessity of recourse to the software on a computer with higher precision to supply the reference values, and permits the testing of software on a computer with any wordlength. To minimize the effect of the local computer environment on the performance of our software testing system, we store the reference function values on tape. The comparison between library function values and the reference values is performed by our Bit Comparison Program designed for highly automated testing [1].

2. The performance of the testing system has been authenticated by a series of rigorous tests that included checks with published tables [1].

3. To avoid conversion errors in the arguments, only exact machine (octal) arguments were used to test the mathematical function subroutines. For the same reason, function values are displayed in octal form. In this way all conversion errors are avoided.

4. Since no theory exists at the present time that would make it possible to construct a set of random arguments that are certain to be representative of the total population of allowable arguments, thus providing a statistical basis for testing, exclusive use of random arguments must currently be considered inadequate for function testing. Special arguments must be used to test certain aspects of performance. It is for this reason that we do not use statistical terms in evaluating the performance of the library. We test functions on a logarithmic scale by supplying arguments in each power of two throughout the entire argument range. Among these are both special and random arguments. Arguments with special bit configurations are designed to probe for weaknesses caused by specific features of the hardware. In view of the testing on a logarithmic scale, the percentages quoted in our breakdown of errors may differ significantly from those given by the manufacturer. Whereas the percentage of full bit accuracy in the various argument ranges is of importance to the tester and the mathematician who does research in algorithms, the user very frequently may not know the range in which his program will generate arguments as a part of a larger calculation. For such a user the most important index of the accuracy of a mathematical function subroutine is its maximum error and this is the index adopted in this study.

5. It is very important to have a clear definition of the error measure used in describing the results of any given performance test. If the definition given in the manufacturer's literature (UP–7876, pp. 2–3, 2–4) were applied to the results of their tests it would show the library in a worse light than it deserves. In fact, interpreting their performance results according to our definition of error measure gives a good agreement with our test results. Therefore, comparison between the manufacturer's test data and ours is made on the basis of our definition of the measure of error. We call our measure of error the "mantissa error." It is the difference resulting from a fixed point subtraction of the mantissa of the reference function value from the mantissa of the library function value. Prior to this subtraction the library function value is normalized to the characteristic of the reference value. The difference is expressed in units of the last bit position (ULP) [3], [4]. For example, an error of 3 ULP implies that the last two binary digits of the library value are in error but not the last three. The mantissa error is equal to the relative error to within a factor depending on the normalization of mantissas in the computer used (a factor of 2 in this case, where mantissas are normalized to lie between 1/2 and 1).

3. Comments on Manufacturer's Documentation

The documentation will be commented upon from two points of view, namely from that of the user and that of the tester [5]. The documentation contained in the FORTRAN V Library manual provides information deemed necessary for the user. It is well written and contains a wealth of information on each routine, including the test data. However, in a few instances the documentation con-

tains inaccuracies. For example, the statement $r^2+s^2<2^{128}$ on p. 2–51 (sec. on CLOG, subsection Argument and Function Range) should read $2^{-129} \leq r^2 + s^2 < 2^{127}$. On p. 2–52 the statement that error termination results if $r^2+s^2>2^{128}$ or $u<2^{-128}$ is incorrect. It should read $r^2+s^2 \geq 2^{127}$ or $r^2 + s^2 = 0$. No error termination results for the real part of the function less than 2^{-128}.

The task of testing would be facilitated by the the identification of mathematical methods and constants with appropriate references, such as Hart et al., Computer Approximations [8]. Furthermore, in listing the functions referenced by a given library routine, a listing of library routines which call the given function would aid in the cross-referencing and the tracing of the effects that errors in a given routine have on the performance of other routines. Modifications of the source code should be reflected in user documentation more promptly. For example, the September 1972 update of the user manual (UP–7876) does not contain mention of the crossover point for small arguments incorporated into the source code DSINCO at least since January 1971.

The source code and the comments contained therein constitute documentation for the tester. They are concerned with the computer implementation of the mathematical algorithms. Two practices, if adopted uniformly throughout the library, would save time and effort in testing and evaluation: The clear identification of all constants in octal—those used to compute function values as well as those used for logical decisions, examples being crossover points and points at which error tracing begins—and the identification at the appropriate point in the coding of the quantities being tested against these latter constants.

4. General Evaluation of the Library

The present library shows that attention has been paid to the choice of good algorithms based on numerical analysis. The computer implementation of these algorithms shows attention to the loss of significance, particularly in the process of argument reduction.[2] Extended precision coding has generally been supplied to avoid loss of accuracy, which requires special programming to circumvent the limitations of hardware (lack of guard digits, finite wordlength, floating point normalization, etc.) and of instruction repertoire (lack of double precision fixed point instruction for multiply and divide, etc.).

It is appropriate at this point to stress the importance of good testing techniques in developing a mathematical library of high quality. Those subroutines that have been tested with the techniques developed at the Jet Propulsion Laboratory [6], namely the single and double precision subroutines [7], exhibit painstaking attention to detail and have fewer programming errors than the complex function routines.

To minimize the effect of hardware limitations, some constants are "fine-tuned" by slightly modifying their bit representation. For example, in the DSINH routine, the octal machine constant 200140 . . . 00, representing unity on lines 93 and 94 of the source code, was changed to 200140 . . . 01 to decrease the number of two bit errors. This technique could be used to a still greater extent, however, and we occasionally make recommendations to that effect in the body of this report. The effect of these recommendations has been explicitly evaluated only for the routines DSINH and DSINCO.

The specifications for most routines in the library are very good. The general philosophy of accepting all arguments for which the function value can be held in the machine is adhered to for the most part. However, the specifications for certain complex functions (CLOG, CCBRT) are unduly restrictive in that the limit of the argument range is based on an overflow condition for $r^2 + s^2$ rather than $(r^2 + s^2)^{1/2}$.

We found the performance of the subroutines to be generally in conformity with the specifications, although, where the maximum error is concerned, the subroutines do not always come up to specifications. One particular example of this concerns the single precision functions, where the specifications give a maximum error of one ULP. This is indeed verified, with the single ex-

[2] This is in marked contrast to its immediate predecessor (March 23, 1966 update of the model 1107 FORTRAN Library Subroutines manual—UP–3947), which like several libraries of different origin contain certain weaknesses due to inadequate numerical analysis, such as the loss of all significance in the calculation of $\sinh x$ for small x by the use of $\sinh x = 1/2(e^x - e^{-x})$.

TABLE 1. *Performance of library functions*

Entry point	No. of arguments	Max error ULP	Percentage of values with ULP error of					Suggested improvements	Programming errors	Comments
			0	1	2	3	4			
			Percent	*Percent*	*Percent*	*Percent*	*Percent*			
SIN	2888	1	98	2				x		
DSIN	8140	2	34	63	3			x		
CSIN	2874							x	x	4–ULP errors due to COS routine.
real		1	97	3						
imag		4	76	20	3	ϵ	ϵ			
COS	2888	4	99	ϵ	ϵ		ϵ	x		Max. error exceeds specs.
DCOS	8140	2	83	12	5			x		
CCOS	2877							x	x	4–ULP errors due to COS routine.
real		4	97	1	ϵ	ϵ	ϵ			
imag		2	78	19		ϵ	4			
TAN	5693	1	95	5					x	
DTAN	8174	4	78	12	8	2	ϵ			
CTAN	2881								x	Error traceable to TANCOT routine.
real		1	96	4						
imag		1	99	1						
COTAN	5678	1	96	4					x	Fails strict monotonicity in range 2^{-26} to 2^{-14}.
DCOTAN	8159	5	57	37	5	1	ϵ		x	Max. error exceeds specs.
ASIN	1839	1	99.7	ϵ						
DASIN	9037	6	54	39	6	1	ϵ			Max. error exceeds specs.
ACOS	1839	1	99.5	ϵ						
DACOS	9037	4	74	23	3	ϵ	ϵ			Max. error exceeds specs.
ATAN	3870	1	96	4						
DATAN	11353	5	68	27	5	ϵ	ϵ			Max. error exceeds specs.
ATAN2(X_1, X_2)	699	1	98	2						The arguments were heavily weighted towards X_1/X_2 very small or very large to test the management function of the entry point.
DATAN2(X_1, X_2)									x	See section on Suggested Improvements, Errors and Specific Comments.
SINH	1384	1	99.7	ϵ						
DSINH	5906	3	8	91	ϵ	ϵ		x		
CSINH	2874							x	x	Max. error exceeds specs.
real		4	76	20	3	ϵ	ϵ			4–ULP errors due to COS routine.
imag		1	97	ϵ						
COSH	1384	1	99.6	ϵ						
DCOSH	5906	2	93	7	ϵ					
CCOSH	2877							x	x	Max. error exceeds specs.
real		4	98.6	1	ϵ	ϵ	ϵ			4–ULP errors due to COS routine.
imag		2	78	19	3					
TANH	1426	1	99	1						
DTANH	5906	1	99	1						

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TABLE 1. *Performance of library functions*—Continued

Entry point	No. of arguments	Max error ULP	Percentage of values with ULP error of					Suggested improvements	Programming errors	Comments
			0	1	2	3	4			
			Percent	*Percent*	*Percent*	*Percent*	*Percent*			
CTANH	2881								x	Max. error exceeds specs.
real		1	99	1						4–ULP errors due to COS routine.
imag		1	96	4						
EXP	2810	1	99	ϵ					x	
DEXP	11812	2	97	3	ϵ					
CEXP	2875							x	x	Max. error exceeds specs.
real		4	98	1	ϵ	ϵ	ϵ			4–ULP errors due to COS routine.
imag		2	97	3	ϵ					
ALOG	1973	1	99.9	ϵ						
DLOG	14706	3	98	1.5	ϵ	ϵ				
CLOG	2101								x	Argument range unnecessarily restrictive.
real		1	85	ϵ						
imag		2	97	3	ϵ					
ALOG10	1973	1	99.3	ϵ						
DLOG10	14706	3	71	28	ϵ	ϵ				The disparity in 1 ULP errors between DLOG and DLOG10 is due to the double precision floating point multiply to convert from base e to base 10.
SQRT	1281	1	90	10						See section on Suggested Improvements, Errors and Specific Comments.
DSQRT	10242	0	100							
CSQRT	3715								x	
real		7+	84	6	ϵ	ϵ	ϵ			
imag		7+	84	5	ϵ	ϵ	ϵ			
CBRT	1537	1	99	1						
DCBRT	12290	2	69	31	ϵ				x	
CCBRT	2106								x	Max. error arbitrarily large. Argument range unnecessarily restrictive.
real			14	23	18	13	7			
imag			16	22	19	13	7			
CABS	3726	1	99.6	ϵ						

ception of the subroutine COS, where a misplaced crossover point results in a maximum error of 4 ULP in an extremely narrow range of the argument. In view of the narrowness of the range it is not surprising that tests based exclusively on random arguments missed this region.

In view of the fact that the operating system in almost every installation is modified to some extent, we would like to point out that such practices as branching on an overflow condition to return a special value of the function (such as is done in DATAN2) may lead to the printing of overflow messages and termination of execution, in a great many computer centers, whether or not such an error tracing capability is supplied by the manufacturer.

5. Tabulation of Results

In this section the results of our testing are presented in tabular form for quick reference. The functions are listed under the entry point name and are in the sequence in which they appear in the manufacturer's manual. A cross-reference table of entry points versus element names follows the tables. The comments on suggested improvements and errors contained in the section entitled Suggested Improvements, Errors and Specific Comments refer primarily to assembly listings and are therefore arranged by element name; the cross-reference table is given to facilitate reference to this later section. The suggested improvements describe for the most part modifications that can be made to the existing algorithms with very little effort and that often lead to substantial improvement in accuracy. They are included because they have come to light as a result of exhaustive testing based on our specialized testing techniques and may be of interest to anyone working on further refinement of these subroutines.

TABLE 2. *Cross-reference table of entry points versus element name*

Entry point	Element name	Entry point	Element name
SIN	SINCO	EXP	EXP
DSIN	DSINCO	DEXP	DEXP
CSIN	CSINCO	CEXP	CEXP
COS	SINCO	ALOG	ALOG
DCOS	DSINCO	DLOG	DLOG
CCOS	CSINCO	CLOG	CLOG
TAN	TANCOT	ALOG10	ALOG
DTAN	DTANCO	DLOG10	DLOG10
CTAN	CTNTNH	SQRT	SQRT
COTAN	TANCOT	DSQRT	DSQRT
DCOTAN	DTANCO	CSQRT	CSQRT
ASIN	ASINCO	CBRT	CBRT
DASIN	DASNCO	DCBRT	DCBRT
ACOS	ASINCO	CCBRT	CCBRT
DACOS	DASNCO	CABS	CABS
ATAN	ATAN		
DATAN	DATAN		
ATAN2(X_1, X_2)	ATAN		
DATAN2(X_1, X_2)	DATAN		
SINH	SINHCO		
DSINH	DSINH		
CSINH	CSINCO		
COSH	SINHCO		
DCOSH	DCOSH		
CCOSH	CSINCO		
TANH	TANH		
DTANH	DTANH		
CTANH	CTNTNH		

6. Suggested Improvements, Errors and Specific Comments

SINCO

Suggested improvements: Relocate crossover point from 2^{-12} to 2^{-13} (line 139 of assembly listing).

Comments: The large maximum error in COS is due to the misplaced crossover point mentioned above.

DSINCO

Suggested improvements:

1. The crossover point at 2^{-944} on line 210 of the assembly listing should be moved to 2^{-30}.
2. The constant on lines 224 and 225 should be changed from 2001400 . . . 0 to 2000777 . .7.
3. The constant on line 229 should be changed from 544067210334 to 544067210335.

The crossover point at 2^{-944} is set too low. This results in computing the function value instead of simply returning the argument for arguments between 2^{-944} and 2^{-30}, leading to increase in execution time and loss of accuracy. The suggested change of the crossover constant 121 to 1743 on line 210 of the assembly listing leads to the following statistics:

	DSIN	DCOS
0 ULP error	85.7 % (6976)	82.7 % (6730)
1 ULP error	12.2 % (995)	12.1 % (986)
2 ULP error	2.1 % (169)	5.2 % (424)

Fine tuning the constant 200140 . . . 0 to the value 200077 . . . 7 is to eliminate under certain conditions the renormalization in the calculation of the cosine. The polynomial representation used above the crossover point to calculate the cosine is of the form $1 - (.308\ldots)x^2 + \ldots$ [8]. When unity is represented by 200140 . . . 0, the result of subtracting the next term in the polynomial must be renormalized no matter how small x is. This brings in a zero in the last bit. Fine tuning avoids this problem for all x. (Note that the reduced argument is less than unity.) The statistics with this change, in addition to the first change, are:

	DSIN	DCOS
0 ULP error	87.6 % (7132)	87.7 % (7136)
1 ULP error	12.2 % (995)	12.2 % (995)
2 ULP error	0.2 % (13)	0.1 % (9)

Changing the constant 544067210334 to 544067210335 is a further step in fine tuning. Changing the constant that represents unity, to avoid renormalization, results in some negative one- and two-ULP errors that can be avoided by decreasing by one bit the magnitude of the coefficient of x^2 in the polynomial representation. The final statistics are as follows:

	DSIN	DCOS
0 ULP error	89.1 % (7254)	89.2 % (7258)
1 ULP error	10.7 % (874)	10.7 % (874)
2 ULP error	0.1 % (12)	0.1 % (8)

CSINCO

Suggested Improvements: Improvement suggested in SINCO will improve the performance of CSINCO.

Errors:

1. Whenever the real or the imaginary part of the function value is in the first bicade (characteristic 000), a zero is returned in place of the correct value for that part of the function. This is caused by testing only the characteristic when checking for underflow.

Comments: The 4–ULP errors come from the SINCO routine.

TANCOT

Errors:

1. The routine returns the incorrect sign for function values whose arguments are in the first seven bicades (characteristics 000 through 006) in TAN.

2. The routine fails to reject the argument 002400000000 for which the correct function value overflows in COTAN. An unnormalized result is returned.

Comments: Routine regularly fails to exhibit strict monotonicity in COTAN in the argument range 2^{-26} through 2^{-14}

DTANCO

Errors:

1. The routine fails to trace overflow values of the function for arguments $0 < x \leq 2^{-1023}$ in DCOTAN.

CTNTNH

Errors:

1. Wrong sign is returned in real part of answer in the first seven bicades (characteristics 000 through 006) in CTAN. The error is traceable to TANCOT routine.

2. Whenever the real or imaginary part of the function value is in the first bicade (characteristic 000), a zero is returned in CTANH in place of the proper value for that part of the function. This error is caused by testing only characteristics when checking for underflow.

DATAN

Errors:

1. The entry point DATAN2(X_1, X_2) has serious programming errors. Whenever $X_2 = 0$ (and $X_1 \neq 0$) the run is terminated due to an illegal operation instead of returning $\pm\frac{1}{2}\pi$.

DSINH

Suggested improvements: The function value should be set equal to the argument whenever the characteristic of the argument is less than or equal to 2^{-30} (characteristic 1743). This would decrease execution time and eliminate most of the 1–ULP errors in this range.

EXP

Errors:

1. The crossover constant to return zero (see lines 71 and 72 of the assembly listing) is placed incorrectly, neglecting to take account of the asymmetry in the storage of positive and negative characteristics. That is, the smallest normalized floating point number is 2^{-129} (octal 000400000000) and the largest is $2^{127}\ (1 - 2^{-27})$, (octal 377777777777).

CEXP

Suggested Improvements: Improvement suggested in SINCO will improve the performance of CEXP.

Errors:

1. Zero is returned for some non-zero function values.

Comments: The 4–ULP errors come from the SINCO routine.

CLOG

Errors:

1. Failure to check for underflow in converting $r^2 + s^2$ from double precision to single precision results in returning erroneous values for the real part of the function.

2. Poor handling of arguments $r^2 + s^2 = 1 + \epsilon$ with ϵ small, prior to passing them to ALOGC$, leads to an arbitrarily large loss of significance in the real part of the function value. These argu-

ments are represented by 34 bits, thus providing 7 guard bits instead of the 127 that are required by the method used here to provide enough significance to comply with the manufacturer's specifications after unity has been subtracted.

SQRT

Comments: The difference between our statistics, which show 90 percent full length accuracy, and those given in the FORTRAN V Library manual (UP−7876), which shows 100 percent full length accuracy, arises from certain of our nonrandom arguments. For example, for arguments of the form $2^{2N}(1 + 2^{-26})$ (where N is an arbitrary integer), i.e., for arguments having the mantissa 400000001, the square root of the argument, when written in the form corresponding to the normalization used, is $2^{N+1}(\frac{1}{2} + 2^{-28} - 2^{-56} + - \ldots)$. In view of the term -2^{-56}, the properly rounded mantissa of the square root is 400000000 rather than 400000001, as given by the library function. However, since our decision on how to round is based on the knowledge of the value in the 56th bit position, we do not consider that the expense of carrying the extra precision is worthwhile.

CSQRT

Errors:

1. For the arguments $(-0, 0)$ and $(0, -0)$ a result of large magnitude is returned with no indication of error.[3] The argument -0 exists because the computer used in the case study is a one's complement machine. The octal representation of $+0$ is 000000000000. The octal representation of -0 is 777777777777.

2. Erroneous results are returned for some arguments with a small characteristic. For example,

Argument		Reference value		Library value	
r:	200434545172	u:	200575625447	u:	200575625447
s:	000777777777	v:	000527235230	v:	377527235230
r:	−020400000001	u:	210400000000	u:	207552023632
s:	220400000001	v:	210400000000	v:	210552023632
r:	375777777777	u:	277552023631	u:	277552023651
s:	−100400000000	v:	−000552023632	v:	−377552023632

DCBRT

Errors:

1. Whenever the characteristic of the argument is 0000, the routine returns wrong results. This is due to failure to check characteristic for zero before subtracting 1 on line 70 of assembly listing.

This work was supported in part by the Institute for Computer Science and Technology of the National Bureau of Standards.

We want to express our deep appreciation for the contribution to this project by A. Liao who wrote several of the extended precision algorithms used in this work and designed tests for several of the functions. We would also like to express our appreciation to David Sookne for advice in some phases of this work.

[3] A similar result was obtained by Jet Propulsion Laboratory (C. L. Lawson, private communication, manufacturer notified).

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(Paper 77B3&4–384)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematical Sciences
Vol. 77B, Nos. 3&4, July–December 1973

Bessel Functions *I* and *J* of Complex Argument and Integer Order

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(June 4, 1973)

A computer program is described for calculating Bessel functions $J_n(z)$ and $I_n(z)$, for z complex, and n a nonnegative integer. The method used is that of backward recursion, with strict control of error, and optimum determination of the point at which to begin the recursion.

Key words: Bessel functions; backward recursion; error bounds; Miller algorithm; difference equation.

1. Method

Given a complex number z and a positive integer NB, BESLCI calculates either

$$I_n(z), \qquad n=0, 1, \ldots, NB-1$$

or

$$J_n(z), \qquad n=0, 1, \ldots, NB-1$$

using double-precision arithmetic. The method used is described in [1][1] and is based on algorithms of Olver [2] and Miller [3], applied to the difference equation

$$y_{n-1}=\frac{2n}{z}\,y_n-\text{SIGN}\cdot y_{n+1}, \tag{1}$$

where SIGN is $+1$ for J's, -1 for I's.

The program sets MAGZ$=[|z|]$, the integer part of $|z|$, $p_{\text{MAGZ}}=0$, $p_{\text{MAGZ}+1}=1$, and then successively calculates

$$p_{n+1}=\text{SIGN}\cdot\left(\frac{2n}{z}\,p_n-p_{n-1}\right), \qquad n=\text{MAGZ}+1,\ \text{MAGZ}+2, \ldots. \tag{2}$$

The sequence is strictly increasing in magnitude [1, sec. 6]. The program takes N to be the least n such that $|p_n|$ exceeds a number TEST defined in section 2 and section 3. It then sets $y_N^{(N)}=0$, $y_{N-1}^{(N)}=1/p_N$, and recurs backward using (1). To the working accuracy, the computed sequence $y_0^{(N)}, y_1^{(N)}, \ldots$ is the recessive solution of (1) which satisfies the boundary condition $y_{\text{MAGZ}}=1$. From this solution, the I's or J's are found by normalizing:

$$J_n(z)=y_n^{(N)}/\mu \qquad n=0, 1, \ldots, NB-1$$

$$I_n(z)=y_n^{(N)}/\mu \qquad n=0, 1, \ldots, NB-1$$

AMS Subject classification: 68A10.
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[1] Figures in brackets indicate literature references at the end of this paper.

where

$$\mu = e^{iz}\left(y_0^{(N)} + 2\sum_{n=1}^{N}(-i)^n y_n^{(N)}\right) \qquad J\text{'s, Im } z > 0$$

$$\mu = y_0^{(N)} + 2\sum_{n=1}^{[N/2]} y_{2n}^{(N)} \qquad J\text{'s, Im } z = 0$$

$$\mu = e^{-iz}\left(y_0^{(N)} + 2\sum_{n=1}^{N} i^N y_n^{(N)}\right) \qquad J\text{'s, Im } z < 0$$

$$\mu = e^{-z}\left(y_0^{(N)} + 2\sum_{n=1}^{N} y_n^{(N)}\right) \qquad I\text{'s, Re } z > 0 \tag{3}$$

$$\mu = y_0^{(N)} + 2\sum_{n=1}^{[N/2]} (-1)^n y_{2n}^{(N)} \qquad I\text{'s, Re } z = 0$$

$$\mu = e^{z}\left(y_0^{(N)} + 2\sum_{n=1}^{N} (-1)^n y_n^{(N)}\right) \qquad I\text{'s, Re } z < 0.$$

2. Error Bounds for the y_n

For $n > \text{MAGZ}$, the truncation error in $y_n^{(N)}$ is

$$T_n^{(N)} = y_n - y_n^{(N)} = p_n \sum_{r=N}^{\infty} \frac{\text{SIGN}^{N-r}}{p_r p_{r+1}},$$

see [2]; equations 5.01 and 5.02. This error is bounded by using the following
LEMMA: *For* $n > MAGZ$, *let*

$$k_n = \frac{p_{n+1}}{p_n} = \frac{2n}{z} - \frac{p_{n-1}}{p_n},$$

and let

$$\lambda_n = \frac{n+1}{|z|} + \sqrt{\left(\frac{n+1}{|z|}\right)^2 - 1}.$$

Let $\rho_n = min\ (|k_n|, \lambda_n)$. *Then for* $m \geq n$, $|k_m| > \rho_n$.

The lemma, for real z, is lemma 2 of [1]. The proof for complex z is essentially the same.

The program insures that

$$\text{TEST}_1 \geq \sqrt{2 \cdot 10^{\text{NSIG}} p_L p_{L+1}}, \tag{4}$$

where $L = \max(\text{MAGZ} + 1, NB - 1)$, and NSIG is the maximum number of significant decimal digits in a double-precision variable on the computer being used. Then N' is the least n such that $|p_n| > \text{TEST}_1$, and N is the least $n \geq N'$ such that

$$|p_n| > \text{TEST} = \sqrt{\frac{\rho_{N'}}{\rho_{N'-1}^2}} \cdot \text{TEST}_1. \tag{5}$$

In consequence of (4) and (5), the relative truncation error $|T_n^{(N)}/y_n|$ is less than $\frac{1}{2} \cdot 10^{-\mathrm{NSIG}}$ for all n in the range $\mathrm{MAGZ} < n \leq L$; see [1, sec. 5].

For $n \leq \mathrm{MAGZ}$, it may be impossible to bound the relative truncation error in the above manner, owing to loss of precision due to cancellation in (1). Experience indicates that this loss is negligible except when the magnitude of $y_n^{(N)}$ oscillates as $n = \mathrm{MAGZ}, \mathrm{MAGZ}-1, \ldots, 0$ in the back-recursion (e.g., calculating J's, with $\mathrm{Re}\ z \gg \mathrm{Im}\ z$). In this case, there will be about D decimals of precision in the values $y_n^{(N)}$, where D is the number of decimals in $J_{\mathrm{MAGZ}}(z)$ $(I_{\mathrm{MAGZ}}(z))$ which corresponds to NSIG significant figures in the same quantity [1, sec. 5].

3. Normalization and Error Bounds for μ

The equations (3) were chosen to keep cancellation under control. Now

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{iz \cos \theta} \cos\ (n\theta) d\theta$$

[4], 9.1.21. The integrand never exceeds $e^{|\mathrm{Im}\ z|}$ in magnitude, so $|J_n(z)| \leq e^{|\mathrm{Im}\ z|}$. Similarly, $|I_n(z)| \leq e^{|\mathrm{Re}\ z|}$. Thus each term of the sums (3) has magnitude less than twice that of the whole sum. In fact, these bounds on the magnitudes of $J_n(z)$ and $I_n(z)$ are rather weak, and cancellation in (3) is less than this would indicate.

Besides bounding the truncation error of the algorithm, the program provides an estimated bound for the truncation error of the normalization sum, defined by eq (3). In the first of these equations, this error is

$$S^{(N)} = e^{iz}\left\{y_0 - y_0^{(N)} + 2 \sum_{n=1}^{\infty} (y_n - y_n^{(N)})\right\}.$$

For $n \leq \mathrm{MAGZ}$, a bound for the error term $y_n - y_n^{(N)}$ is unavailable. For $\mathrm{MAGZ} < n < N$, $|y_n - y_n^{(N)}| \leq p_n \rho_n / (\rho_n^2 - 1)$; see [1, sec. 5]. To avoid storing all the p_n, the program allows only for terms for which $n \geq N$. Here $y_n^{(N)} = 0$, and

$$y_n = p_n \sum_{r=n}^{\infty} \frac{1}{p_r p_{r+1}}.$$

Therefore,

$$|y_n^{(N)} - y_n| \leq \left|\frac{1}{p_{n+1}}\right| \left\{1 + \left|\frac{p_n}{p_{n+2}}\right| + \left|\frac{p_n}{p_{n+2}} \frac{p_{n+1}}{p_{n+3}}\right| + \ldots\right\}$$

$$\leq \frac{1}{|p_{n+1}|}\left|1 + \frac{1}{\rho_n^2} + \frac{1}{\rho_n^4} + \ldots\right| \leq \frac{\rho_{N'}^2}{(\rho_{N'}^2 - 1)\,|p_{n+1}|},$$

compare section 2 above. Now let

$$R^{(N)} = 2 \sum_{n=N}^{\infty} |y_n|.$$

Then

$$R^{(N)} \leq \frac{2\rho_{N'}^2}{\rho_{N'}^2 - 1} \sum_{n=N}^{\infty} \left|\frac{1}{p_{n+1}}\right| \leq \frac{2\rho_{N'}^2}{(\rho_{N'}^2 - 1)|p_{N+1}|}\left(1 + \frac{1}{\rho_{N'}} + \frac{1}{\rho_{N'}^2} + \ldots\right)$$

$$\leq \frac{2\rho_{N'}^2}{(\rho_{N'}^2 - 1)\,(\rho_{N'} - 1)\,|p_N|}.$$

The program sets $\mathrm{TEST}_1 \geq 2 \cdot 10^{\mathrm{NSIG}}$. The normalization factor μ is $e^{-iz}/J_{\mathrm{MAGZ}}(z)$ [1, sec. 5], so

$$\left|\frac{R^{(N)}}{\mu}\right| = \left|J_{\mathrm{MAGZ}}(z)\right| \cdot \frac{R^{(N)}}{e^{\mathrm{Im}\, z}} \leq R^{(N)}$$

$$\leq \frac{2\rho_{N'}^2}{(\rho_{N'}+1)(\rho_{N'}-1)^2|p_N|} \leq \frac{2\rho_{N'}^2}{(\rho_{N'}+1)(\rho_{N'}-1)^2} \cdot \frac{1}{2} \cdot 10^{-\mathrm{NSIG}}. \tag{6}$$

The bound (6) holds for the first, third, fourth and sixth equations of (3); the derivations are the same. Similarly, the second and fifth equations yield

$$\left|\frac{R^{(N)}}{\mu}\right| \leq \frac{2\rho_{N'}^3}{(\rho_{N'}^2-1)^2} \cdot \frac{1}{2} \cdot 10^{-\mathrm{NSIG}}.$$

These bounds are rather weak, and the error $|S^{(N)}/\mu|$ turns out to be less than $\frac{1}{2} \cdot 10^{-\mathrm{NSIG}}$.

4. References

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(Paper 77B3&4–385)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematical Sciences
Vol. 77B, Nos. 3&4, July–December 1973

Certification of an Algorithm for Bessel Functions of Real Argument

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(June 4, 1973)

The accuracy of routine BESLRI is certified.

Key words: Bessel functions; bit comparison; error bounds.

1. Introduction

Bessel function values $I_n(x)$ and $J_n(x)$ (x real), generated by BESLRI [1],[1] were compared with check values. For $|x| < 64$, these check values were calculated via the ascending series

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!(n+k)!}, \qquad I_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(x^2/4)^k}{k!(n+k)!},$$

using multiprecision arithmetic [2]. For $x \geq 64$, the asymptotic formulas for large argument were used:

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{(\mu-1)}{8x} + \frac{(\mu-1)(\mu-9)}{2!(8x)^2} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \cdots\right)$$

where $\mu = 4n^2$, and error is controlled by 7.16 of [3], and

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\{P(n, x)\cos\chi - Q(n, x)\sin\chi\}$$

where $\chi = x - \left(\frac{n}{2} + \frac{1}{4}\right)\pi$, and

$$P(n, x) \sim 1 - \frac{(\mu-1)(\mu-9)}{2!(8x)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8x)^4} + \cdots$$

$$Q(n, x) \sim \frac{\mu-1}{8x} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \cdots,$$

AMS Subject Classification: 68A10.
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[1] Figures in brackets indicate the literature references at the end of this paper.

with errors bounded by the first neglected term [4, 9.2.9 and 9.2.10]. These were programmed in double precision. For $x \leq -64$, $-x$ was used as the argument, and the sign of the function value was determined from the equations

$$J_n(-x) = (-1)^n J_n(x), \qquad I_n(-x) = (-1)^n I_n(x).$$

The method used was that of bit comparison; the difference between the 60-bit mantissas of test and check values was expressed as a multiple, m, say, of the last bit. Such a bit error corresponds to a relative error between $m \cdot 2^{-60}$ and $m \cdot 2^{-59}$.

This test is too strict near a zero of $J_n(x)$, where absolute error is a more realistic measure. Thus for $n < |x|$, both mantissas were right-shifted so that the bit error described above was a multiple of the 60th binary place. These cases are given above the dashed line in the computer printout; statistics for this test are compiled separately.

A bit comparison test was also used to check the section of the code involving the two-term ascending series for small $|x|$.

The error return feature was tested exactly as in [5].

2. The Bit Comparison Test

To test the function $I_n(x)$, ten arguments were chosen in each interval

$$A_j = \{x : 2^{j-1} \leq |x| < 2^j\}, \qquad j = -13(1)10,$$

the 60-bit mantissas of each x being produced by a pseudorandom number generator [5]. The orders used in all cases were $n = 0(1)15$. The only modification occurred in the set A_{10}, for which $|x|$ was limited to 700 to avoid computer overflow in the final answers. Summary statistics for the 60-bit test on I_n are:

373 results correct to 60 bits
583 results in error by 1 multiple of the last bit
383 results in error by 2 multiples of the last bit
288 results in error by 3 multiples of the last bit
232 results in error by 4 multiples of the last bit
196 results in error by 5 multiples of the last bit
176 results in error by 6 multiples of the last bit
130 results in error by 7 multiples of the last bit
1479 results in error by more than 7 multiples of the last bit

3840 (total)

The largest bit error was $(36)_8$, implying that at least 55 bits are correct in each case.

For the function $J_n(x)$, the set of intervals A_j was expanded to $j = -13(1)16$. Again, ten pseudorandom values were used in each interval, and the orders used were 0(1)15. Summaries for the tests are:

Test on 60 Significant Bits $|x| < 64$

403 results correct to 60 significant bits
643 results in error by 1 multiple of the last bit
453 results in error by 2 multiples of the last bit
331 results in error by 3 multiples of the last bit

232 results in error by 4 multiples of the last bit
171 results in error by 5 multiples of the last bit
109 results in error by 6 multiples of the last bit
68 results in error by 7 multiples of the last bit
82 results in error by more than 7 multiples of the last bit

2492 (total)

The largest bit error was $(15)_8$, showing that at least 56 bits are correct in each case.

Test on 60 Binary Places $|x| < 64$

161 results correct to 60 places
200 results in error by 1 multiple of the last place
100 results in error by 2 multiples of the last place
85 results in error by 3 multiples of the last place
56 results in error by 4 multiples of the last place
46 results in error by 5 multiples of the last place
45 results in error by 6 multiples of the last place
14 results in error by 7 multiples of the last place
1 result in error by more than 7 multiples of the last place.

708 (total)

The largest bit error was $(10)_8$, showing that 56 binary places are correct in each case, and 57 places in all but one case.

In the test on 60 binary places for $|x| \geq 64$, the largest error was $(422)_8$. This occurred in $J_5(x)$ and $J_7(x)$ for $x = 59766.6\ldots$. Bearing in mind that about 60,000 back-recursion steps were executed, that each step included two multiplications and one addition, and also that the computer uses truncation in floating-point arithmetic rather than roundoff, we see that the error of $(422)_8 = 274$ is quite reasonable. The other errors can be explained similarly.

For the test of the two-term ascending series, arguments of 2^{-16} and 2^{-100} were used; the former is just barely small enough for the truncation error to be neglected. Both $I_n(x)$ and $J_n(x)$ were tested with $n = 0(1)15$, and the largest error was 6 multiples of the last bit.

3. Testing of Error Return

The method used is the same as in [5]. Arguments used were $x = 2^k$ where $k = -13(1)9$ for I's, and $k = -13(1)13$ for J's. In the test on Bessel function values of order NCALC−1, the largest error was 6 multiples of the last bit; most were zero.

In the test on order $[x]$, the largest error was $(46)_8$, which occurred in $J_{8192}(8192)$. This test value involved about 1600 more back-recursion steps than were required for the check value, so this error is quite reasonable. The other errors were $(35)_8$ for $x = 4096$, zero for $x = 2048$, $(12)_8$ for $x = 1024$; for $x \leq 512$, the errors did not exceed 5.

4. Summary and Conclusions

For $n = 0(1)15$ and the values of x tested, BESLRI yielded $J_n(x)$ and $I_n(x)$ with either a relative error or an absolute error bounded by 10^{-14}. The latter assessment applied to $J_n(x)$ when $n < |x|$. For $|x| < 64$, the bound was 10^{-16}. The error return and ascending series features were tested and found to be executed accurately in all cases.

For other values of n and x, similar accuracy can be expected, since the tests we have used include large and small values of $|x|$ and $\frac{|x|}{n}$, the former as large as 2^{16}.

Calculations were performed on a UNIVAC 1108 under EXEC 2. All test values were calculated with double precision mantissa of 60 bits, corresponding to just over 18 figures. Subroutine executions took approximately 0.2 N milliseconds, where

$$N = \max\ (|x|, \mathrm{NB}) + 10.$$

5. Appendix: Algorithm BESLRI

```
@ ELT BESLCI,1,730112, 51748

000001            SUBROUTINE BESLCI( X,Y,NB,IZE,BR,BI,NCALC)                    00010
000002          C THIS ROUTINE CALCULATES BESSEL FUNCTIONS I AND J OF           00020
000003          C COMPLEX ARGUMENT AND INTEGER ORDER.                           00030
000004          C                                                               00040
000005          C                                                               00050
000006          C       EXPLANATION OF VARIABLES IN THE CALLING SEQUENCE        00060
000007          C                                                               00070
000008          C X     DOUBLE PRECISION REAL PART OF THE COMPLEX ARGUMENT      00080
000009          C       FOR WHICH I•S OR J•S ARE TO BE CALCULATED.  IF I•S      00090
000010          C       ARE TO BE CALCULATED, ABS(X) MUST NOT EXCEED EXPARG     00100
000011          C       (WHICH SEE BELOW).                                      00110
000012          C Y     IMAGINARY PART OF THE ARGUMENT.  IF J•S ARE TO BE       00120
000013          C       CALCULATED, ABS(Y) MUST NOT EXCEED EXPARG.              00130
000014          C NB    INTEGER TYPE.  1 + HIGHEST ORDER TO BE CALCULATED.      00140
000015          C       IT MUST BE POSITIVE.                                    00150
000016          C IZE   INTEGER TYPE.  ZERO IF J•S ARE TO BE CALCULATED, 1      00160
000017          C       IF I•S ARE TO BE CALCULATED.                            00170
000018          C BR    DOUBLE PRECISION VECTOR OF LENGTH NB, NEED NOT BE       00180
000019          C       INITIALIZED BY USER.  IF THE ROUTINE TERMINATES         00190
000020          C       NORMALLY, (NCALC=NB), IT RETURNS THE REAL PART OF       00200
000021          C       J(OR I)-SUB-ZERO THROUGH J(OR I)-SUB-NB-MINUS-ONE       00210
000022          C       OF Z IN THIS VECTOR.                                    00220
000023          C BI    IMAGINARY ANALOG OF BR.                                 00230
000024          C NCALC INTEGER TYPE, NEED NOT BE INITIALIZED BY USER.          00240
000025          C       BEFORE USING THE RESULTS, THE USER SHOULD CHECK THAT    00250
000026          C       NCALC=NB, I.E. ALL ORDERS HAVE BEEN CALCULATED TO       00260
000027          C       THE DESIRED ACCURACY.  SEE ERROR RETURNS BELOW.         00270
000028          C                                                               00280
000029          C                                                               00290
000030          C       EXPLANATION OF MACHINE-DEPENDENT CONSTANTS              00300
000031          C                                                               00310
000032          C NSIG  DECIMAL SIGNIFICANCE DESIRED.  SHOULD BE SET TO         00320
000033          C       IFIX(ALOG10(2)•NBIT+1), WHERE NBIT IS THE NUMBER OF     00330
000034          C       BITS IN THE MANTISSA OF A DOUBLE PRECISION VARIABLE.    00340
000035          C       SETTING NSIG HIGHER WILL INCREASE CPU TIME WITHOUT       00350
000036          C       INCREASING ACCURACY, WHILE SETTING NSIG LOWER WILL       00360
000037          C       DECREASE ACCURACY.  IF ONLY SINGLE-PRECISION             00370
000038          C       ACCURACY IS DESIRED, REPLACE NBIT BY THE NUMBER OF       00380
000039          C       BITS IN THE MANTISSA OF A SINGLE-PRECISION VARIABLE.     00390
000040          C       THE RELATIVE TRUNCATION ERROR IS LIMITED TO T•.5•10      00400
000041          C       ••-NSIG FOR ORDER GREATER THAN ABS(Z), AND FOR ORDER     00410
000042          C       LESS THAN ABS(Z) (GENERAL TEST), THE RELATIVE ERROR      00420
000043          C       IS LIMITED TO T FOR FUNCTION VALUES OF MAGNITUDE AT      00430
000044          C       LEAST 1, AND THE ABSOLUTE ERROR IS LIMITED TO T FOR      00440
000045          C       SMALLER VALUES.                                          00450
000046          C NTEN  LARGEST INTEGER K SUCH THAT 10••K IS MACHINE-            00460
000047          C       REPRESENTABLE IN DOUBLE PRECISION.                       00470
000048          C LARGEZ UPPER LIMIT ON THE MAGNITUDE OF Z.  BEAR IN MIND        00480
000049          C       THAT IF ABS(Z)=N, THEN AT LEAST N ITERATIONS OF THE      00490
000050          C       BACKWARD RECURSION WILL BE EXECUTED.                     00500
000051          C EXPARG LARGEST DOUBLE PRECISION ARGUMENT THAT THE LIBRARY      00510
000052          C       DEXP ROUTINE CAN HANDLE.                                 00520
000053          C                                                                00530
000054          C                                                                00540
```

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```
000055          C                               ERROR RETURNS                                   00550
000056          C                                                                               00560
000057          C       LET G DENOTE EITHER I OR J.                                             00570
000058          C       IN CASE OF AN ERROR, NCALC.NE.NB, AND NOT ALL G*S                       00580
000059          C  ARE CALCULATED TO THE DESIRED ACCURACY.                                      00590
000060          C       IF NCALC.LT.0, AN ARGUMENT IS OUT OF RANGE.  NB.LE.0                    00600
000061          C  OR IZE IS NEITHER 0 NOR 1 OR IZE=0 AND ABS(Y).GT.EXPARG,                     00610
000062          C  OR IZE=1 AND ABS(X).GT.EXPARG.  IN THIS CASE, THE VECTORS                    00620
000063          C  BR AND BI ARE NOT CALCULATED, AND NCALC IS SET TO                            00630
000064          C  MINO(NB,0)-1  SO NCALC.NE.NB.                                               00640
000065          C       NB.GT.NCALC.GT.0 WILL OCCUR IF NB.GT.MAGZ AND ABS(G-                    00650
000066          C  SUB-NB-OF-Z/G-SUB-MAGX-OF-Z).LT.10.**(NTEN/2), I. E. NB                      00660
000067          C  IS MUCH GREATER THAN MAGZ.  IN THIS CASE, BR(N) AND BI(N)                    00670
000068          C  ARE CALCULATED TO THE DESIRED ACCURACY FOR N.LE.NCALC,                       00680
000069          C  BUT FOR NCALC.LT.N.LE.NB, PRECISION IS LOST.  IF N.GT.                       00690
000070          C  NCALC AND ABS(G(NCALC-1)/G(N-1)).EQ.10**-K, THEN THE LAST                    00700
000071          C  K SIGNIFICANT FIGURES OF G(N-1) (=BR(N)*I=BI(N)) ARE                         00710
000072          C  ERRONEOUS.  IF THE USER WISHES TO CALCULATE G(N-1) TO                        00720
000073          C  HIGHER ACCURACY, HE SHOULD USE AN ASYMPTOTIC FORMULA FOR                     00730
000074          C  LARGE ORDER.                                                                 00740
000075          C                                                                               00750
000076                DOUBLE PRECISION                                                          00760
000077               1 X,Y,BR,BI,PR,PI,PLASTR,PLASTI,POLDR,POLDI,PSAVER,                        00770
000078               2 PSAVEI,EXPARG,TEST,TOVER,TEMPAR,TEMPAI,TEMPBR,TEMPBI,                   00780
000079               3 TEMPCR,TEMPCI,SIGN,SUMR,SUMI,ZINVR,ZINVI                                00790
000080                DIMENSION BR(NB),BI(NB)                                                   00800
000081                DATA NSIG,NTEN,LARGEZ,EXPARG/19,307,10000,7.D2/                           00810
000082                TEMPAR=DSQRT(X*X+Y*Y)                                                     00820
000083                MAGZ=IFIX(SNGL(TEMPAR))                                                   00830
000084                IF(NB.GT.0.AND.MAGZ.LE.LARGEZ.AND.((IZE.EQ.0.AND.                         00840
000085               1 DABS(Y).LE.EXPARG).OR.(IZE.EQ.1.AND.DABS(X).LE.                          00850
000086               2 EXPARG))) GO TO 1                                                        00860
000087          C ERROR RETURN -- Z, NB, OR IZE IS OUT OF RANGE                                 00870
000088                NCALC=MINO(NB,0)-1                                                        00880
000089                RETURN                                                                    00890
000090             1  SIGN=DBLE(FLOAT(1-2*IZE))                                               00900
000091                NCALC=NB                                                                  00910
000092          C USE 2-TERM ASCENDING SERIES FOR SMALL Z                                       00920
000093                IF(TEMPAR**4.LT..1D0**NSIG) GO TO 50                                      00930
000094          C INITIALIZE THE CALCULATION OF THE P=S                                         00940
000095                NBMZ=NB-MAGZ                                                              00950
000096                N=MAGZ+1                                                                  00960
000097                IF(DABS(X).LT.DABS(Y)) GO TO 2                                            00970
000098                ZINVR=1.D0/(X+Y*Y/X)                                                      00980
000099                ZINVI=-Y*ZINVR/X                                                          00990
000100                GO TO 3                                                                   01000
000101             2  ZINVI=-1.D0/(Y+X*X/Y)                                                   01010
000102                ZINVR=-X*ZINVI/Y                                                          01020
000103             3  PLASTR=1.D0                                                            01030
000104                PLASTI=0.D0                                                               01040
000105                PR=SIGN*DBLE(FLOAT(2*N))*ZINVR                                            01050
000106                PI=SIGN*DBLE(FLOAT(2*N))*ZINVI                                            01060
000107                TEST=2.D0*1.D1**NSIG                                                      01070
000108                M=0                                                                       01080
000109                IF(NBMZ.LT.3) GO TO 6                                                     01090
000110          C CALCULATE P=S UNTIL N=NB-1.  CHECK FOR POSSIBLE OVERFLOW.                     01100
000111                TOVER=1.D1**(NTEN-NSIG)                                                   01110
000112                NSTART=MAGZ+2                                                             01120
```

```
000113                NEND=NB-1                                                 01130
000114                DO 5 N=NSTART,NEND                                        01140
000115                  POLDR=PLASTR                                            01150
000116                  POLDI=PLASTI                                            01160
000117                  PLASTR=PR                                               01170
000118                  PLASTI=PI                                               01180
000119                  PR=SIGN*(DBLE(FLOAT(2*N))*(PLASTR*ZINVR-PLASTI*ZINVI    01190
000120                 1  )-POLDR)                                             01200
000121                  PI=SIGN*(DBLE(FLOAT(2*N))*(PLASTI*ZINVR+PLASTR*ZINVI    01210
000122                 1  )-POLDI)                                             01220
000123                  IF((PR/TOVER)**2+(PI/TOVER)**2-1.D0) 5,5,7              01230
000124                5   CONTINUE                                              01240
000125                  N=NEND                                                  01250
000126           C CALCULATE SPECIAL SIGNIFICANCE TEST FOR NBMZ.GT.2.            01260
000127                  TEMPBI=DMAX1(DABS(PR),DABS(PI))                         01270
000128                  TEMPBI=TEMPBI*DSQRT(2.D0+1.D1**NSIG*DSQRT(((PR/TEMPBI)  01280
000129                 1**2+(PI/TEMPBI)**2)*((PLASTR/TEMPBI)**2+(PLASTI/        01290
000130                 2TEMPBI)**2)))                                           01300
000131                  TEST=DMAX1(TEST,TEMPBI)                                 01310
000132           C CALCULATE P*S UNTIL SIGNIFICANCE TEST IS PASSED.              01320
000133                6 N=N+1                                                   01330
000134                  POLDR=PLASTR                                            01340
000135                  POLDI=PLASTI                                            01350
000136                  PLASTR=PR                                               01360
000137                  PLASTI=PI                                               01370
000138                  PR=SIGN*(DBLE(FLOAT(2*N))*(PLASTR*ZINVR-PLASTI*ZINVI)   01380
000139                 1 -POLDR)                                                01390
000140                  PI=SIGN*(DBLE(FLOAT(2*N))*(PLASTI*ZINVR+PLASTR*ZINVI)   01400
000141                 1 -POLDI)                                                01410
000142                  IF((PR/TEST)**2+(PI/TEST)**2.LT.1.D0) GO TO 6           01420
000143                  IF(N.EQ.1) GO TO 12                                     01430
000144           C CALCULATE STRICT VARIANT OF SIGNIFICANCE TEST, AND            01440
000145           C CALCULATE P*S UNTIL THIS TEST IS PASSED.                      01450
000146                  N=1                                                     01460
000147                  TEMPBI=DMAX1(DABS(PR),DABS(PI))                         01470
000148                  TEMPBR=DSQRT((PR/TEMPBI)**2+(PI/TEMPBI)**2)/            01480
000149                 1 ((PLASTR/TEMPBI)**2+(PLASTI/TEMPBI)**2))               01490
000150                  TEMPBI=DBLE(FLOAT(N+1))/TEMPAR                          01500
000151                  IF(TEMPBR*1.D0/TEMPBR.GT.2.D0+TEMPBI) TEMPBR=TEMPBI     01510
000152                 1 +DSQRT(TEMPBI**2-1.D0)                                 01520
000153                  TEST=TEST/DSQRT(TEMPBR-1.D0/TEMPBR)                     01530
000154                  IF((PR/TEST)**2+(PI/TEST)**2-1.D0) 6,12,12              01540
000155                7 NSTART=N+1                                              01550
000156           C TO AVOID OVERFLOW, NORMALIZE P*S BY DIVIDING BY TOVER.        01560
000157           C CALCULATE P*S UNTIL UNNORMALIZED P WOULD OVERFLOW.            01570
000158                  PR=PR/TOVER                                             01580
000159                  PI=PI/TOVER                                             01590
000160                  PLASTR=PLASTR/TOVER                                     01600
000161                  PLASTI=PLASTI/TOVER                                     01610
000162                  PSAVER=PR                                               01620
000163                  PSAVEI=PI                                               01630
000164                  TEMPCR=PLASTR                                           01640
000165                  TEMPCI=PLASTI                                           01650
000166                  TEST=1.D1**(2*NSIG)                                     01660
000167                8 N=N+1                                                   01670
000168                  POLDR=PLASTR                                            01680
000169                  POLDI=PLASTI                                            01690
000170                  PLASTR=PR                                               01700
```

```
000171                PLASTI=PI                                                 01710
000172                PR=SIGN*( DBLE( FLOAT( 2*N ))*( PLASTR*ZINVR-PLASTI*ZINVI )  01720
000173               1 -POLDR)                                                  01730
000174                PI=SIGN*( DBLE( FLOAT( 2*N ))*( PLASTI*ZINVR+PLASTR*ZINVI )  01740
000175               1 -POLDI )                                                 01750
000176                IF( PR**2+PI**2.LE.TEST) GO TO 8                          01760
000177           C CALCULATE BACKWARD TEST, AND FIND NCALC, THE HIGHEST N        01770
000178           C SUCH THAT THE TEST IS PASSED.                                 01780
000179                TEMPBR=DSQRT( (PLASTR**2+PLASTI**2)/(POLDR**2+POLDI**2)   01790
000180               1 )                                                        01800
000181                TEMPBI=DBLE(FLOAT( N ))/TEMPAR                            01810
000182                IF( TEMPBR+1.D0/TEMPBR.GT.2.D0+TEMPBI ) TEMPBR=TEMPBI+    01820
000183               1 DSQRT( TEMPBI**2-1.D0 )                                  01830
000184                TEST=.5D0*( 1.D0-1.D0/TEMPBR**2)/1.D1**NSIG               01840
000185                TEST=(( PLASTR**2+PLASTI**2)*TEST)*((POLDR**2+POLDI**2)   01850
000186               1 *TEST)                                                   01860
000187                PR=PLASTR*TOVER                                           01870
000188                PI=PLASTI*TOVER                                           01880
000189                N=N-1                                                     01890
000190                NEND=MIN0( NB,N)                                          01900
000191                DO 9 NCALC=NSTART,NEND                                    01910
000192                  POLDR=TEMPCR                                            01920
000193                  POLDI=TEMPCI                                            01930
000194                  TEMPCR=PSAVER                                           01940
000195                  TEMPCI=PSAVEI                                           01950
000196                  PSAVER=SIGN*( DBLE( FLOAT( 2*N ))*( TEMPCR*ZINVR-TEMPCI*  01960
000197               1  ZINVI )-POLDR)                                         01970
000198                  PSAVEI=SIGN*( DBLE( FLOAT( 2*N ))*( TEMPCI*ZINVR+TEMPCR*  01980
000199               1  ZINVI )-POLDI )                                        01990
000200                  IF(( PSAVER**2+PSAVEI**2)*( TEMPCR**2+TEMPCI**2)-TEST)  02000
000201               1  9,9,10                                                 02010
000202               9  CONTINUE                                               02020
000203                  NCALC=NEND+1                                            02030
000204               10 NCALC=NCALC-1                                           02040
000205           C THE COEFFICIENT OF B(N) IN THE NORMALIZATION SUM IS           02050
000206           C M*SQRT( -1 )**IMAG, WHERE M=-2,0, OR 2, AND IMAG IS 0 OR 1.   02060
000207           C CALCULATE RECURSION RULES FOR M AND IMAG, AND INITIALIZE      02070
000208           C THEM.                                                         02080
000209               12 N=N+1                                                    02090
000210                  TEMPBR=DBLE( FLOAT( IZE ))*X+DBLE( FLOAT( 1-IZE ))*Y    02100
000211                  IPOS=0                                                  02110
000212                  IF( TEMPBR ) 13,14,13                                   02120
000213               13 IPOS=IFIX( SNGL( 1.1D0*TEMPBR/DABS( TEMPBR )))          02130
000214               14 MRECUR=4*(( 2*IZE*IPOS)/2)-3-2*( IZE*IPOS)              02140
000215                  K=2*IPOS*2+IZE-IPOS**2-IZE                              02150
000216                  L=N-4*( N/4 )                                           02160
000217                  MLAST=2*8*(( K+L )/4 )-4*(( K+L )/2 )                   02170
000218                  IF( IPOS.EQ.0.AND.( L.EQ.1.OR.L.EQ.3)) MLAST=0          02180
000219                  L=L*3-4*(( L*3 )/4 )                                    02190
000220                  M     =2*8*(( K+L )/4 )-4*(( K+L )/2 )                  02200
000221                  IF( IPOS.EQ.0.AND.( L.EQ.1.OR.L.EQ.3)) M=0              02210
000222                  IMRECR=( 1-IZE )*IPOS**2                                02220
000223                  IMAG=IMRECR*( L-2*( L/2 ))                              02230
000224           C INITIALIZE THE BACKWARD RECURSION AND THE NORMALIZATION        02240
000225           C SUM.                                                          02250
000226                  TEMPBR=0.D0                                             02260
000227                  TEMPBI=0.D0                                             02270
000228                  IF( DABS( PI ).GT.DABS( PR )) GO TO 15                  02280
```


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```
      TEMPAR=1.D0/(PR+PI*(PI/PR))
      TEMPAI=-(PI*TEMPAR)/PR
      GO TO 16
   15 TEMPAI=-1.D0/(PI+PR*(PR/PI))
      TEMPAR=-(PR*TEMPAI)/PI
   16 IF(IMAG.NE.0) GO TO 17
      SUMR=DBLE(FLOAT(M))*TEMPAR
      SUMI=DBLE(FLOAT(M))*TEMPAI
      GO TO 18
   17 SUMR=-DBLE(FLOAT(M))*TEMPAI
      SUMI=DBLE(FLOAT(M))*TEMPAR
   18 NEND=N-NB
      IF(NEND) 26,22,19
C RECUR BACKWARD VIA DIFFERENCE EQUATION CALCULATING (BUT
C NOT STORING) BR(N) AND BI(N) UNTIL N=NB.
   19 DO 21 L=1,NEND
        N=N-1
        TEMPCR=TEMPBR
        TEMPCI=TEMPBI
        TEMPBR=TEMPAR
        TEMPBI=TEMPAI
        PR=DBLE(FLOAT(2*N))*ZINVR
        PI=DBLE(FLOAT(2*N))*ZINVI
        TEMPAR=PR*TEMPBR-PI*TEMPBI-SIGN*TEMPCR
        TEMPAI=PR*TEMPBI+PI*TEMPBR-SIGN*TEMPCI
        IMAG=(1-IMAG)*IMRECR
        K=MLAST
        MLAST=M
        M=K*MRECUR
        IF(IMAG.NE.0) GO TO 20
        SUMR=SUMR+DBLE(FLOAT(M))*TEMPAR
        SUMI=SUMI+DBLE(FLOAT(M))*TEMPAI
        GO TO 21
   20   SUMR=SUMR-DBLE(FLOAT(M))*TEMPAI
        SUMI=SUMI+DBLE(FLOAT(M))*TEMPAR
   21   CONTINUE
C STORE BR(NB), BI(NB)
   22 BR(N)=TEMPAR
      BI(N)=TEMPAI
      IF(N.GT.1) GO TO 23
C NB=1.  SINCE 2*TEMPAR AND 2*TEMPAI WERE ADDED TO SUMR AND
C SUMI RESPECTIVELY, WE MUST SUBTRACT TEMPAR AND TEMPAI
      SUMR=SUMR-TEMPAR
      SUMI=SUMI-TEMPAI
      GO TO 35
C CALCULATE AND STORE BR(NB-1),BI(NB-1)
   23 N=N-1
      PR=DBLE(FLOAT(2*N))*ZINVR
      PI=DBLE(FLOAT(2*N))*ZINVI
      BR(N)=PR*TEMPAR-PI*TEMPAI-SIGN*TEMPBR
      BI(N)=PR*TEMPAI+PI*TEMPAR-SIGN*TEMPBI
      IF(N.EQ.1) GO TO 34
      IMAG=(1-IMAG)*IMRECR
      K=MLAST
      MLAST=M
      M=K*MRECUR
      IF(IMAG.NE.0) GO TO 24
      SUMR=SUMR+DBLE(FLOAT(M))*BR(N)
```

```
000287                SUMI=SUMI+DBLE(FLOAT(M))*BI(M)                          02870
000288                GO TO 30                                                02880
000289             24 SUMR=SUMR-DBLE(FLOAT(M))*BI(M)                          02890
000290                SUMI=SUMI+DBLE(FLOAT(M))*BR(M)                          02900
000291                GO TO 30                                                02910
000292          C N.LT.NB, SO STORE BR(N), BI(N), AND SET HIGHER ORDERS ZERO  02920
000293             26 BR(N)=TEMPAR                                            02930
000294                BI(N)=TEMPAI                                            02940
000295                NEND=-NEND                                              02950
000296                DO 27 L=1,NEND                                          02960
000297                  BR(N+L)=0.D0                                          02970
000298             27   BI(N+L)=0.D0                                         02980
000299             30 NEND=N-2                                                02990
000300                IF(NEND.EQ.0) GO TO 33                                  03000
000301          C CALCULATE VIA DIFFERENCE EQUATION AND STORE BR(N),BI(N),     03010
000302          C UNTIL N=2                                                   03020
000303                DO 32 L=1,NEND                                          03030
000304                  N=N-1                                                 03040
000305                  PR=DBLE(FLOAT(2*N))*ZINVR                             03050
000306                  PI=DBLE(FLOAT(2*N))*ZINVI                             03060
000307                  BR(N)=PR*BR(N+1)-PI*BI(N+1)-SIGN*BR(N+2)              03070
000308                  BI(N)=PR*BI(N+1)+PI*BR(N+1)-SIGN*BI(N+2)              03080
000309                  IMAG=(1-IMAG)*IMRECR                                  03090
000310                  K=NLAST                                               03100
000311                  NLAST=N                                               03110
000312                  N=K+NRECUR                                            03120
000313                  IF(IMAG.NE.0) GO TO 31                                03130
000314                  SUMR=SUMR+DBLE(FLOAT(N))*BR(N)                        03140
000315                  SUMI=SUMI+DBLE(FLOAT(N))*BI(N)                        03150
000316                  GO TO 32                                              03160
000317             31   SUMR=SUMR-DBLE(FLOAT(N))*BI(N)                        03170
000318                  SUMI=SUMI+DBLE(FLOAT(N))*BR(N)                        03180
000319             32   CONTINUE                                              03190
000320          C CALCULATE AND STORE BR(1), BI(1)                            03200
000321             33 BR(1)=2.D0*(BR(2)*ZINVR-BI(2)*ZINVI)-SIGN*BR(3)         03210
000322                BI(1)=2.D0*(BR(2)*ZINVI+BI(2)*ZINVR)-SIGN*BI(3)         03220
000323             34 SUMR=SUMR+BR(1)                                         03230
000324                SUMI=SUMI+BI(1)                                         03240
000325          C CALCULATE NORMALIZATION FACTOR, TEMPAR +I*TEMPAI             03250
000326             35 IF(IZE.EQ.1) GO TO 36                                   03260
000327                TEMPCR=DBLE(FLOAT(IPOS))*Y                              03270
000328                TEMPCI=DBLE(FLOAT(-IPOS))*X                             03280
000329                GO TO 37                                                03290
000330             36 TEMPCR=DBLE(FLOAT(IPOS))*X                              03300
000331                TEMPCI=DBLE(FLOAT(IPOS))*Y                              03310
000332             37 TEMPCR=DEXP(TEMPCR)                                     03320
000333                TEMPBR=DCOS(TEMPCI)                                     03330
000334                TEMPBI=DSIN(TEMPCI)                                     03340
000335                IF(DABS(SUMR).LT.DABS(SUMI)) GO TO 38                   03350
000336                TEMPCI=SUMI/SUMR                                        03360
000337                TEMPCR=(TEMPCR/SUMR)/(1.D0+TEMPCI*TEMPCI)               03370
000338                TEMPAR=TEMPCR*(TEMPBR+TEMPBI*TEMPCI)                    03380
000339                TEMPAI=TEMPCR*(TEMPBI-TEMPBR*TEMPCI)                    03390
000340                GO TO 39                                                03400
000341             38 TEMPCI=SUMR/SUMI                                        03410
000342                TEMPCR=(TEMPCR/SUMI)/(1.D0+TEMPCI*TEMPCI)               03420
000343                TEMPAR=TEMPCR*(TEMPBR+TEMPCI*TEMPBI)                    03430
000344                TEMPAI=TEMPCR*(TEMPBI+TEMPCI-TEMPBR)                    03440

000345          C NORMALIZE                                                   03450
000346             39 DO 40 N=1,NB                                            03460
000347                  TEMPBR=DR(N)*TEMPAR-BI(N)*TEMPAI                      03470
000348                  BI(N)=BR(N)*TEMPAI+BI(N)*TEMPAR                       03480
000349             40   BR(N)=TEMPBR                                         03490
000350                RETURN                                                  03500
000351          C TWO-TERM ASCENDING SERIES FOR SMALL Z                       03510
000352             50 TEMPAR=1.D0                                             03520
000353                TEMPAI=0.D0                                             03530
000354                TEMPCR=.25D0*(X*X-Y*Y)                                  03540
000355                TEMPCI=.5D0*X*Y                                         03550
000356                BR(1)=1.D0-SIGN*TEMPCR                                  03560
000357                BI(1)=-SIGN*TEMPCI                                      03570
000358                IF(NB.EQ.1) GO TO 52                                    03580
000359                DO 51 N=2,NB                                            03590
000360                  TEMPBR=(TEMPAR*X-TEMPAI*Y)/DBLE(FLOAT(2*N-2))         03600
000361                  TEMPAI=(TEMPAR*Y+TEMPAI*X)/DBLE(FLOAT(2*N-2))         03610
000362                  TEMPAR=TEMPBR                                         03620
000363                  TEMPBR=DBLE(FLOAT(N))                                 03630
000364                  BR(N)=TEMPAR*(1.D0-SIGN*TEMPCR/TEMPBR)                03640
000365                1 +TEMPAI*TEMPCI/TEMPBR                                 03650
000366             51   BI(N)=TEMPAI*(1.D0-SIGN*TEMPCR/TEMPBR)                03660
000367                1 -TEMPAR*TEMPCI/TEMPBR                                 03670
000368             52 RETURN                                                  03680
000369                END                                                     03690

      END CUR LCC 1102-0038 L8
```

6. References

[1] Algorithm BESLRI, see previous paper.

[2] Maximon, Leonard C., FORTRAN Program for Arbitrary Precision Arithmetic, unpublished data.

[3] Olver, Frank W. J., Error Bounds for Asymptotic Expansion with an Application to Cylinder Functions of Large Argument, in Asymptotic Solutions of Differential Equations and Their Applications, Calvin H. Wilcox, Editor (Wiley & Sons, 1964).

[4] National Bureau of Standards, Handbook of Mathematical Functions, Nat. Bur. Stand. (U.S.), Appl. Math. Ser. 55 (1964).

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(Paper 77B3&4–386)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77A, Nos. 3&4, July–December 1973

Bessel Functions of Real Argument and Integer Order

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(June 4, 1973)

A computer program is described for calculating Bessel Functions $J_n(x)$ and $I_n(x)$, for x real, and n a nonnegative integer. The method used is that of backward recursion, with strict control of error.

Key words: Backward recursion; Bessel functions; difference equation; error bounds; Miller algorithms.

1. Method

Given a real number x and a positive integer NB, BESLRI calculates either

$$I_n(x), \qquad n = 0, 1, \ldots, \mathrm{NB} - 1$$

or

$$J_n(x), \qquad n = 0, 1, \ldots, \mathrm{NB} - 1$$

using double-precision arithmetic. The method, which is described in [1],[2] is based on algorithms of Olver [2] and Miller [3], applied to the difference equation

$$y_{n-1} = \frac{2n}{x}\, y_n - \mathrm{SIGN} \cdot y_{n+1} \tag{1}$$

where SIGN is $+1$ for J's, -1 for I's.

The program sets $\mathrm{MAGX} = [\,|x|\,]$, the integer part of $|x|$, $p_{\mathrm{MAGX}} = 0$, $p_{\mathrm{MAGX}+1} = 1$, and then successively calculates

$$p_{n+1} = \frac{2n}{|x|}\, p_n - \mathrm{SIGN} \cdot p_{n-1} \qquad n = \mathrm{MAGX} + 1, \mathrm{MAGX} + 2, \ldots \tag{2}$$

The sequence is strictly increasing. The program takes N to be the least n such that p_n exceeds a number TEST defined in sections 2 and 3. It then sets $y_N^{(N)} = 0$, $y_{N-1}^{(N)} = 1/p_N$, and recurs backward using (1). The computed sequence $y_0^{(N)}, y_1^{(N)}, \ldots$ is the recessive solution of (1) which satisfies the boundary condition $y_{\mathrm{MAGX}} = 1$. From this solution, the I's and J's are found by normalizing

$$J_n(x) = y_n^{(N)}/\mu \qquad n = 0, 1, \ldots, \mathrm{NB} - 1$$

$$I_n(x) = y_n^{(N)}/\mu \qquad n = 0, 1, \ldots, \mathrm{NB} - 1$$

AMS Subject Classification: 68A10.

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[2] Figures in brackets indicate the literature references at the end of this paper.

where

$$\mu = y_0^{(N)} + 2\sum_{k=1}^{[N/2]} y_{2k}^{(N)} \qquad \text{for } J\text{'s}$$

$$\mu = \left(y_0^{(N)} + 2\sum_{k=1}^{[N/2]} y_{2k}^{(N)}\right) \Big/ \cosh x \qquad \text{for } I\text{'s.} \tag{3}$$

2. Error Bounds for the y_n

For $n > \text{MAGX}$, the truncation error in $y_n^{(N)}$ is

$$T_n^{(N)} = y_n - y_n^{(N)} = p_n \sum_{r=N}^{\infty} \frac{1}{p_r p_{r+1}} ;$$

see [2], eqs (5.01) and (5.02). This error is bounded by using Lemma 2 of [1], which states that for $s \geq r$, $\rho_s = p_{s+1}/p_s \geq \min\,(p_{r+1}/p_r,\ (r+1)/|x| + \sqrt{(r+1)^2/|x|^2 - 1})$.

The program sets

$$\text{TEST}_1 \geq \sqrt{2 \cdot 10^{\text{NSIG}} p_L p_{L+1}}, \tag{4}$$

where $L = \max(\text{MAGX} + 1,\ \text{NB} - 1)$, and NSIG is the maximum number of significant decimal digits in a double-precision variable on the computer being used. Then N' is the least n such that $p_n > \text{TEST}_1$. For J's, N is the least $n \geq N'$ such that

$$p_n > \text{TEST} = \sqrt{\frac{\rho_{N'}}{\rho_{N'}^2 - 1}} \cdot \text{TEST}_1. \tag{5}$$

In consequence of (4) and (5), the relative truncation error $|\, T_n^{(N)} / y_n \,|$ is less than $\frac{1}{2} \cdot 10^{-\text{NSIG}}$ for all n in the range $\text{MAGX} < n \leq L$; see [1, sec. 5]. For I's, N is taken to be N'. Here a glance at (1) shows $p_n / p_{n-1} > 2$ for all $n > \text{MAGX}$, so the proof given in [1, sec. 5] for J's may also be used for I's, with ρ_N replaced by 2 throughout.

In eq (1) for I's, $\frac{2n}{x}\, y_n^{(N)}$ and $-\text{SIGN} \cdot y_{n+1}^{(N)}$ have the same sign since if $x > 0$, then $y_n^{(N)} > 0$ for all n, and if $x < 0$, then consecutive $y_n^{(N)}$'s have opposite signs. Thus relative errors $< \epsilon$ in $y_{n+1}^{(N)}$ and $y_n^{(N)}$ will produce a relative error $< \epsilon$ in $y_{n-1}^{(N)}$. Therefore, the relative truncation error in all $I_n(x)$, $0 \leq x < \text{NB}$, is bounded by $\frac{1}{2} \cdot 10^{-\text{NSIG}}$.

For $J_n(x)$, $n \leq \text{MAGX}$, the relative truncation error cannot be bounded, since as $n = \text{MAGX}$, $\text{MAGX} - 1, \ldots, 1$ in (1), the values $J_n(x)$ oscillate, and precision is lost owing to cancellation. In this range, $J_n(x)$ is accurate to about D decimal places, where D is the number of decimals in $J_{\text{MAGX}+1}(x)$ which corresponds to NSIG significant figures in the same quantity [1, sec. 5].

3. Normalization and Error Bounds for μ

The eqs (3) were chosen to keep cancellation under control. First,

$$|\, J_0(x) \,| \leq 1, \quad |\, J_n(x) \,| \leq \frac{1}{\sqrt{2}}, \quad \text{for } n \geq 1 \tag{6}$$

[4, 2.5], so each term of the sum (3) for J's is less than $\sqrt{2}$ times the whole sum. Since (6) is a rather weak bound, cancellation is even less than this would indicate.

For I's, cancellation is avoided altogether, since all terms in the sum (3) for I's have the same sign, as shown in section 2.

Besides bounding the truncation error of the algorithm, the program provides an estimated bound for the truncation error of the normalization sum, defined by eqs (3). For J's, this error is

$$S^{(N)} = y_0 - y_0^{(N)} + 2 \sum_{k=1}^{\infty} (y_{2k} - y_{2k}^{(N)}).$$

For $2k \leq \text{MAGX}$, a bound for the error term $y_{2k}^{(N)} - y_{2k}$ is unavailable. For $\text{MAGX} < 2k < N$, $| y_{2k}^{(N)} - y_{2k} | < p_{2k}\rho_N/(\rho_N^2 - 1)$ [1, sec. 5]. To avoid storing all the p_{2k}, the program allows only for terms for which $2k \geq N$. Here $y_{2k}^{(N)} = 0$, and $y_{2k} = p_{2k} \sum_{r=2k}^{N} 1/(p_r p_{r+1})$. Therefore

$$| y_{2k}^{(N)} - y_{2k} | = \left| \frac{1}{p_{2k+1}} \left\{ 1 + \frac{p_{2k}}{p_{2k+2}} + \frac{p_{2k}\ p_{2k+1}}{p_{2k+2}\ p_{2k+3}} + \cdots \right\} \right|$$

$$\leq \frac{1}{p_{2k+1}} \left\{ 1 + \frac{1}{\rho_{2k}^2} + \frac{1}{\rho_{2k}^4} + \cdots \right\} = \frac{\rho_{N'}^2}{(\rho_{N'}^2 - 1) p_{2k+1}};$$

compare section 2 above. Now let

$$R^{(N)} = \sum_{k = [\frac{N+1}{2}]}^{\infty} | y_{2k} |.$$

Then

$$R^{(N)} \leq \frac{\rho_{N'}^2}{\rho_{N'}^2 - 1} \sum_{k = [\frac{N+1}{2}]}^{\infty} \frac{1}{p_{2k+1}} \leq \frac{\rho_{N'}^2}{(\rho_{N'}^2 - 1) p_{N+1}} \left\{ 1 + \frac{1}{\rho_{N'}^2} + \frac{1}{\rho_{N'}^4} + \cdots \right\}$$

$$\leq \frac{\rho_{N'}^3}{(\rho_{N'}^2 - 1)^2 p_N}. \tag{7}$$

For J's the program sets $\text{TEST}_1 \geq 2 \cdot 10^{\text{NSIG}}$. The normalization factor μ is $1/J_{\text{MAGX}}(x)$ [1, sec. 5], so

$$\left| \frac{R^{(N)}}{\mu} \right| = \left| J_{\text{MAGX}}(x) \right| R^{(N)} \leq \frac{| J_{\text{MAGX}}(x) |\ \rho_{N'}^3}{(\rho_{N'}^2 - 1)^2} \cdot \frac{1}{2} \cdot 10^{-\text{NSIG}}.$$

This is a rather weak upper bound, and the error $\left| \frac{S^{(N)}}{\mu} \right|$ turns out to be less than $\frac{1}{2} \cdot 10^{-\text{NSIG}}$.

For I's, $p_n/p_{n-1} > 2$ so the analog of (7) is

$$R^{(N)} \leq \frac{8}{9 p_N \cosh x};$$

this is derived by substituting 2 for $\rho_{N'}$ in (7), and introducing the factor cosh x; see (3).

By setting $\mathrm{TEST}_1 \geq \dfrac{2 \cdot 10^{-\mathrm{NSIG}}}{\exp(0.461 \cdot \mathrm{MAGX})}$, the program insures that

$$R^{(N)} \leq \frac{4 \exp(0.461 \cdot \mathrm{MAGX})}{9 \cdot 10^{-\mathrm{NSIG}} \cdot \cosh x}.$$

Here, μ is $\dfrac{1}{I_{\mathrm{MAGX}}}(x)$, so the relative error $|R^{(N)}/\mu|$ is bounded by

$$\left| \frac{R^{(N)}}{\mu} \right| \leq \frac{4 I_{\mathrm{MAGX}}(x) \exp(0.461 \cdot \mathrm{MAGX})}{9 \cosh x \cdot 10^{-\mathrm{NSIG}}}. \tag{8}$$

Now Kapteyn's inequality [4, 8.7] states that

$$|J_n(nz)| \leq \left| \frac{z \exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right|^n.$$

Let $z = \dfrac{|x|}{\mathrm{MAGX}}$, so for large x, $z = 1$, approximately.

Then

$$|I_{\mathrm{MAGX}}(x)| = |J_{\mathrm{MAGX}}(ix)| = |J_{\mathrm{MAGX}}(\mathrm{MAGX} \cdot iz)|$$

$$\leq \left| \frac{z \exp\sqrt{1+z^2}}{1+\sqrt{1+z^2}} \right|^{\mathrm{MAGX}}.$$

For large x, the approximate bound for $I_{\mathrm{MAGX}}(x)$ is

$$\left(\frac{\exp\sqrt{2}}{1+\sqrt{2}} \right)^{\mathrm{MAGX}} < \exp(0.533 \cdot \mathrm{MAGX}).$$

Substituting this in (8), we obtain

$$\left| \frac{R^{(N)}}{\mu} \right| < \frac{4 \cdot \exp(0.533\ \mathrm{MAGX}) \cdot \exp(0.461 \cdot \mathrm{MAGX})}{9 \cosh x \cdot 10^{-\mathrm{NSIG}}}$$

$$< \frac{4}{9} \cdot 10^{-\mathrm{NSIG}} < \frac{1}{2} \cdot 10^{-\mathrm{NSIG}}.$$

Besides this approximate upper bound, the strict bounds

$$\left| \frac{R^{(N)}}{\mu} \right| < 7.6 \cdot 10^{-\mathrm{NSIG}} \qquad |x| \geq 1$$

and

$$\left| \frac{R^{(N)}}{\mu} \right| < 0.565 \cdot 10^{-\mathrm{NSIG}} \qquad |x| < 1$$

may be derived by using Kapteyn's inequality to obtain a strict bound for $|I_{\mathrm{MAGX}+1}(x)|$.

4. Appendix: Algorithm BESLRI

```
@ ELT BESLRI,1,730222, 58730

000001                SUBROUTINE BESLRI(X,NB,IZE,B,NCALC)                    00000100
000002          C THIS ROUTINE CALCULATES BESSEL FUNCTIONS I AND J OF REAL    00000200
000003          C ARGUMENT AND INTEGER ORDER.                                 00000300
000004          C                                                             00000400
000005          C                                                             00000500
000006          C       EXPLANATION OF VARIABLES IN THE CALLING SEQUENCE      00000600
000007          C                                                             00000700
000008          C X     DOUBLE PRECISION REAL ARGUMENT FOR WHICH I*S OR J*S   00000800
000009          C       ARE TO BE CALCULATED.  IF I*S ARE TO BE CALCULATED,   00000900
000010          C       ABS(X) MUST BE LESS THAN EXPARG (WHICH SEE BELOW).    00001000
000011          C NB    INTEGER TYPE.  1 + HIGHEST ORDER TO BE CALCULATED.    00001100
000012          C       IT MUST BE POSITIVE.                                  00001200
000013          C IZE   INTEGER TYPE.  ZERO IF J*S ARE TO BE CALCULATED, 1    00001300
000014          C       IF I*S ARE TO BE CALCULATED.                          00001400
000015          C B     DOUBLE PRECISION VECTOR OF LENGTH NB, NEED NOT BE     00001500
000016          C       INITIALIZED BY USER.  IF THE ROUTINE TERMINATES       00001600
000017          C       NORMALLY (NCALC=NB), IT RETURNS J(OR I)-SUB-ZERO      00001700
000018          C       THROUGH J(OR I)-SUB-NB-MINUS-ONE OF X IN THIS         00001800
000019          C       VECTOR.                                               00001900
000020          C NCALC INTEGER TYPE, NEED NOT BE INITIALIZED BY USER.        00002000
000021          C       BEFORE USING THE RESULTS, THE USER SHOULD CHECK THAT  00002100
000022          C       NCALC=NB, I.E. ALL ORDERS HAVE BEEN CALCULATED TO     00002200
000023          C       THE DESIRED ACCURACY.  SEE ERROR RETURNS BELOW.       00002300
000024          C                                                             00002400
000025          C                                                             00002500
000026          C     EXPLANATION OF MACHINE-DEPENDENT CONSTANTS              00002600
000027          C                                                             00002700
000028          C NSIG  DECIMAL SIGNIFICANCE DESIRED.  SHOULD BE SET TO       00002800
000029          C       IFIX(ALOG10(2)*NBIT+1), WHERE NBIT IS THE NUMBER OF   00002900
000030          C       BITS IN THE MANTISSA OF A DOUBLE PRECISION VARIABLE.  00003000
000031          C       SETTING NSIG LOWER WILL RESULT IN DECREASED ACCURACY  00003100
000032          C       WHILE SETTING NSIG HIGHER WILL INCREASE CPU TIME      00003200
000033          C       WITHOUT INCREASING ACCURACY.  THE TRUNCATION ERROR    00003300
000034          C       IS LIMITED TO T=.5*10**-NSIG FOR J*S OF ORDER LESS    00003400
000035          C       THAN ARGUMENT, AND TO A RELATIVE ERROR OF T FOR       00003500
000036          C       I*S AND THE OTHER J*S.                                00003600
000037          C NTEN  LARGEST INTEGER K SUCH THAT 10**K IS MACHINE-         00003700
000038          C       REPRESENTABLE IN DOUBLE PRECISION.                    00003800
000039          C LARGEX UPPER LIMIT ON THE MAGNITUDE OF X.  BEAR IN MIND     00003900
000040          C       THAT IF ABS(X)=N, THEN AT LEAST N ITERATIONS OF THE   00004000
000041          C       BACKWARD RECURSION WILL BE EXECUTED.                  00004100
000042          C EXPARG LARGEST DOUBLE PRECISION ARGUMENT THAT THE LIBRARY   00004200
000043          C       DEXP ROUTINE CAN HANDLE.                              00004300
000044          C                                                             00004400
000045          C                                                             00004500
000046          C                    ERROR RETURNS                            00004600
000047          C                                                             00004700
000048          C       LET G DENOTE EITHER I OR J.                           00004800
000049          C       IN CASE OF AN ERROR, NCALC.NE.NB, AND NOT ALL G*S     00004900
000050          C  ARE CALCULATED TO THE DESIRED ACCURACY.                    00005000
000051          C       IF NCALC.LT.0, AN ARGUMENT IS OUT OF RANGE.  NB.LE.0  00005100
000052          C  OR IZE IS NEITHER 0 NOR 1 OR IZE=1 AND ABS(X).GE.EXPARG.   00005200
000053          C  IN THIS CASE, THE B-VECTOR IS NOT CALCULATED, AND NCALC    00005300
000054          C  IS SET TO MIN0(NB,0)-1 SO NCALC.NE.NB.                     00005400
```

```
C       NB.GT.NCALC.GT.0 WILL OCCUR IF NB.GT.MAGX AND ABS(G-
C  SUB-NB-OF-X/G-SUB-MAGX-OF-X).LT.10.**(NTEN/2), I. E. NB
C  IS MUCH GREATER THAN MAGX.  IN THIS CASE, B(N) IS CALCU-
C  LATED TO THE DESIRED ACCURACY FOR N.LE.NCALC, BUT FOR
C  NCALC.LT.N.LE.NB, PRECISION IS LOST.  IF N.GT.NCALC AND
C  ABS(B(NCALC)/B(N)).EQ.10**K, THEN ONLY THE FIRST NSIG-K
C SIGNIFICANT FIGURES OF B(N) MAY BE TRUSTED.  IF THE USER
C WISHES TO CALCULATE B(N) TO HIGHER ACCURACY, HE SHOULD USE
C AN ASYMPTOTIC FORMULA FOR LARGE ORDER.
C
      DOUBLE PRECISION
     1 X,B,P,TEST,TEMPA,TEMPB,TEMPC,EXPARG,SIGN,SUM,TOVER,
     2 PLAST,POLD,PSAVE,PSAVEL
      DIMENSION B(NB)
      DATA NSIG,NTEN,LARGEX,EXPARG/19,307,100000,7.D2/
      TEMPA=DABS(X)
      MAGX=IFIX(SNGL(TEMPA))
      IF(NB.GT.0.AND.MAGX.LE.LARGEX.AND.(IZE.EQ.0.OR.
     1(IZE.EQ.1.AND.TEMPA.LE.EXPARG))) GO TO 1
C ERROR RETURN -- X,NB,OR IZE IS OUT OF RANGE
      NCALC=MIN0(NB,0)-1
      RETURN
    1 SIGN=DBLE(FLOAT(1-2*IZE))
      NCALC=NB
C USE 2-TERM ASCENDING SERIES FOR SMALL X
      IF(TEMPA**4.LT..1D0**NSIG) GO TO 30
C INITIALIZE THE CALCULATION OF P*S
      NBMX=NB-MAGX
      N=MAGX+1
      PLAST=1.D0
      P=DBLE(FLOAT(2*N))/TEMPA
C CALCULATE GENERAL SIGNIFICANCE TEST
      TEST=2.D0*1.D1**NSIG
      IF(IZE.EQ.1.AND.2*MAGX.GT.5*NSIG) TEST=DSQRT(TEST*P)
      IF(IZE.EQ.1.AND.2*MAGX.LE.5*NSIG)TEST=TEST/1.585**MAGX
      M=0
      IF(NBMX.LT.3) GO TO 4
C CALCULATE P*S UNTIL N=NB-1.  CHECK FOR POSSIBLE OVERFLOW.
      TOVER=1.D1**(NTEN-NSIG)
      NSTART=MAGX+2
      NEND=NB-1
      DO 3 N=NSTART,NEND
        POLD=PLAST
        PLAST=P
        P=DBLE(FLOAT(2*N))*PLAST/TEMPA-SIGN*POLD
        IF(P-TOVER) 3,3,5
    3   CONTINUE
      N=NEND
C CALCULATE SPECIAL SIGNIFICANCE TEST FOR NBMX.GT.2.
      TEST=DMAX1(TEST,DSQRT(PLAST*1.D1**NSIG)*DSQRT(2.D0*P))
C CALCULATE P*S UNTIL SIGNIFICANCE TEST PASSES
    4 N=N+1
      POLD=PLAST
      PLAST=P
      P=DBLE(FLOAT(2*N))*PLAST/TEMPA-SIGN*POLD
      IF(P.LT.TEST) GO TO 4
      IF(IZE.EQ.1.OR.M.EQ.1) GO TO 12
C FOR J*S, A STRONG VARIANT OF THE TEST IS NECESSARY.
```

```
000113          C CALCULATE IT, AND CALCULATE P*S UNTIL THIS TEST IS PASSED.          00011300
000114                M=1                                                           00011400
000115                TEMPB=P/PLAST                                                 00011500
000116                TEMPC=DBLE(FLOAT(N+1))/TEMPA                                  00011600
000117                IF(TEMPB+1.D0/TEMPB.GT.2.D0*TEMPC)TEMPB=TEMPC+DSQRT           00011700
000118               1(TEMPC**2-1.D0)                                               00011800
000119                TEST=TEST/DSQRT(TEMPB-1.D0/TEMPB)                             00011900
000120                IF(P-TEST) 4,12,12                                            00012000
000121          C TO AVOID OVERFLOW, DIVIDE P*S BY TOVER.  CALCULATE P*S            00012100
000122          C UNTIL ABS(P).GT.1.                                                00012200
000123             5 TOVER=1.D1**NTEN                                               00012300
000124               P=P/TOVER                                                     00012400
000125               PLAST=PLAST/TOVER                                             00012500
000126               PSAVE=P                                                       00012600
000127               PSAVEL=PLAST                                                  00012700
000128               NSTART=N+1                                                    00012800
000129             6 N=N+1                                                         00012900
000130               POLD=PLAST                                                    00013000
000131               PLAST=P                                                       00013100
000132               P=DBLE(FLOAT(2*N))*PLAST/TEMPA-SIGN*POLD                      00013200
000133               IF(P.LE.1.D0) GO TO 6                                         00013300
000134               TEMPB=DBLE(FLOAT(2*N))/TEMPA                                  00013400
000135               IF(IZE.EQ.1) GO TO 8                                          00013500
000136               TEMPC=.5D0*TEMPB                                              00013600
000137               TEMPB=PLAST/POLD                                              00013700
000138               IF(TEMPB+1.D0/TEMPB.GT.2.D0*TEMPC)TEMPB=TEMPC+DSQRT           00013800
000139              1(TEMPC**2-1.D0)                                               00013900
000140          C CALCULATE BACKWARD TEST, AND FIND NCALC, THE HIGHEST N            00014000
000141          C SUCH THAT THE TEST IS PASSED.                                     00014100
000142             8 TEST=.5D0*POLD*PLAST*(1.D0-1.D0/TEMPB**2)/1.D1**NSIG           00014200
000143               P=PLAST*TOVER                                                 00014300
000144               N=N-1                                                         00014400
000145               NEND=MIN0(NB,N)                                               00014500
000146               DO 9 NCALC=NSTART,NEND                                        00014600
000147                 POLD=PSAVEL                                                 00014700
000148                 PSAVEL=PSAVE                                                00014800
000149                 PSAVE=DBLE(FLOAT(2*N))*PSAVEL/TEMPA-SIGN*POLD               00014900
000150                 IF(PSAVE*PSAVEL-TEST) 9,9,10                                00015000
000151             9   CONTINUE                                                    00015100
000152               NCALC=NEND+1                                                  00015200
000153            10 NCALC=NCALC-1                                                 00015300
000154          C THE SUM B(1)+2B(3)+2B(5)... IS USED TO NORMALIZE.  M, THE         00015400
000155          C COEFFICIENT OF B(N), IS INITIALIZED TO 2 OR 0.                     00015500
000156            12 N=N+1                                                        00015600
000157               M=2*N-4*(N/2)                                                 00015700
000158          C INITIALIZE THE BACKWARD RECURSION AND THE NORMALIZATION           00015800
000159          C SUM                                                             00015900
000160               TEMPB=0.D0                                                    00016000
000161               TEMPA=1.D0/P                                                  00016100
000162               SUM=DBLE(FLOAT(M))*TEMPA                                      00016200
000163               NEND=N-NB                                                     00016300
000164               IF(NEND) 17,15,13                                             00016400
000165          C RECUR BACKWARD VIA DIFFERENCE EQUATION, CALCULATING (BUT         00016500
000166          C NOT STORING) B(N), UNTIL N=NB.                                    00016600
000167            13 DO 14 L=1,NEND                                               00016700
000168               N=N-1                                                         00016800
000169               TEMPC=TEMPB                                                   00016900
000170               TEMPB=TEMPA                                                   00017000
```

```
000171                TEMPA=(DBLE(FLOAT(2*N))*TEMPB)/X-SIGN*TEMPC                00017100
000172                M=2-M                                                      00017200
000173             14   SUM=SUM+DBLE(FLOAT(M))*TEMPA                            00017300
000174          C STORE B(NB)                                                   00017400
000175             15 B(N)=TEMPA                                                00017500
000176                IF(NB.GT.1) GO TO 16                                       00017600
000177          C NB=1.  SINCE 2*TEMPA WAS ADDED TO THE SUM, TEMPA MUST BE      00017700
000178          C SUBTRACTED                                                    00017800
000179                SUM=SUM-TEMPA                                            00017900
000180                GO TO 23                                                 00018000
000181          C CALCULATE AND STORE B(NB-1)                                   00018100
000182             16 N=N-1                                                    00018200
000183                B(N) =(DBLE(FLOAT(2*N))*TEMPA)/X-SIGN*TEMPB              00018300
000184                IF(N.EQ.1) GO TO 22                                      00018400
000185                M=2-M                                                    00018500
000186                SUM=SUM+DBLE(FLOAT(M))*B(N)                              00018600
000187                GO TO 19                                                 00018700
000188          C N.LT.NB, SO STORE B(N) AND SET HIGHER ORDERS TO ZERO          00018800
000189             17 B(N)=TEMPA                                               00018900
000190                NENJ=NEND                                                00019000
000191                DO 18 L=1,NENJ                                           00019100
000192             18   B(N+L)=0.D0                                            00019200
000193             19 NEND=N-2                                                 00019300
000194                IF(NEND.EQ.0) GO TO 21                                   00019400
000195          C CALCULATE VIA DIFFERENCE EQUATION AND STORE B(N),             00019500
000196          C UNTIL N=2                                                     00019600
000197                DO 20 L=1,NENJ                                           00019700
000198                  N=N-1                                                  00019800
000199                  B(N)=(DBLE(FLOAT(2*N))*B(N+1))/X-SIGN*B(N+2)           00019900
000200                  M=2-M                                                  00020000
000201             20   SUM=SUM+DBLE(FLOAT(M))*B(N)                            00020100
000202          C CALCULATE B(1)                                                00020200
000203             21 B(1)=2.D0*B(2)/X-SIGN*B(3)                                00020300
000204             22 SUM=SUM+B(1)                                              00020400
000205          C NORMALIZE--IF IZE=1, DIVIDE SUM BY COSH(X),  DIVIDE ALL       00020500
000206          C B(N) BY SUM.                                                  00020600
000207             23 IF(IZE.EQ.0) GO TO 25                                     00020700
000208                TEMPA=DEXP(DABS(X))                                      00020800
000209                SUM=2.D0*SUM/(TEMPA+1.D0/TEMPA)                          00020900
000210             25 DO 26 N=1,NB                                             00021000
000211             26   B(N)=B(N)/SUM                                          00021100
000212                RETURN                                                   00021200
000213          C TWO-TERM ASCENDING SERIES FOR SMALL X                         00021300
000214             30 TEMPA=1.D0                                               00021400
000215                TEMPB=-.2500*X*X*SIGN                                    00021500
000216                B(1)=1.D0+TEMPB                                          00021600
000217                IF(NB.EQ.1) GO TO 32                                     00021700
000218                DO 31 N=2,NB                                             00021800
000219                  TEMPA=TEMPA*X/DBLE(FLOAT(2*N-2))                       00021900
000220             31   B(N)=TEMPA*(1.D0+TEMPB/DBLE(FLOAT(N)))                 00022000
000221             32 RETURN                                                   00022100
000222                END                                                      00022200
    3.

      END CUR LCC 1102-0038 L8
```

5. References

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(Paper 77B3&4–387)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, No. 3, July–September 1973

Certification of an Algorithm for Bessel Functions of Complex Argument

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(June 4, 1973)

Computer tests of algorithm BESLCI are described, and the results of the test are given.

Key words: Backward recursion; Bessel function; bit comparison; error bounds; relative error.

1. Introduction

Bessel function values $I_n(z)$, $J_n(z)$ generated by BESLCI [1] were compared with check values calculated via the ascending series

$$J_n(z)=\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!(n+k)!}, \qquad I_n(z)=\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(n+k)!},$$

using multiprecision arithmetic [2].[2] The 60-bit mantissa of each part of the test value was subtracted from the corresponding mantissa of the check value, and the result expressed as a multiple, m, say, of the last bit. For most errors mentioned in this certification, $|m| \leq 7$. Bit comparison was used in preference to calculation of relative error in order to simplify computations. An error of m bits in the mantissa corresponds to a relative error between $m \cdot 2^{-60}$ and $m \cdot 2^{-59}$.

This test is too strict, however, near a zero of the real or imaginary part of the function being tested. For complex values of z, it is more realistic to compare the bit error with the *greater* of the real and imaginary parts, since the relative error in a complex number $\xi+i\eta$ is $(\delta\xi+i\delta\eta)/|\xi+i\eta|$, $\delta\xi$ and $\delta\eta$ denoting the respective errors in the real and imaginary parts. For arguments in the neighborhoods of zeroes of Re $J_n(z)$, Im $J_n(z)$, Re $I_n(z)$, or Im $I_n(z)$, an assessment of absolute errors was made to 60 binary places by right-shifting the mantissa of both test and check values before subtraction. This was done whenever the order n was less than $|z|$ and the modulus of the check value less than $\frac{1}{2}$. These cases are distinguished by asterisks in the computer printout; statistics for this test are compiled separately.

A bit comparison test was also used to check the section of code involving the two-term ascending series for small $|z|$.

A second feature of BESLCI which was tested systematically is the error return. This portion of the code determines whether all orders $n=0, 1, \ldots, \mathrm{NB}-1$ can be calculated to the desired accuracy. A modification occurs if NB is so large compared with $|z|$ that there is underflow in the representation of $J_{\mathrm{NB}-1}(z)$ or $I_{\mathrm{NB}-1}(z)$. When this happens, the program determines the largest

AMS Subject Classification: 68A10.

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[2] Figures in brackets indicate the literature references at the end of this paper.

value NCALC ($>|z|$) such that accurate values can be calculated for $n=0, 1, \ldots, \text{NCALC}-1$; these values are then calculated. To test this, BESLCI was called with NB = 5600. In each case, the program returned NCALC < NB, insuring that the error return loop had been executed. Then the values of NCALC and the Bessel function of order NCALC−1 were checked by calling BESLCI with NB set equal to NCALC; this involved another execution of the error return loop. In each case, the new NCALC was identical with the old. Finally, the Bessel function value of order $[|z|]$ was checked by calling BESLCI with $\text{NB} = [|z|] + 1$; this involved no execution of the error return loop. In both cases, bit comparisons on 60 significant bits were performed.

Results of the tests are described in the next two sections.

2. The Bit Comparison Test

Arguments z were chosen as follows. Let R_j denote the set of rational numbers $2^j \cdot p$, where p is a 60-bit fraction, with $\frac{1}{2} \leq p < 1$. Two complex arguments were generated in each of the sets

$$A_j = \{x + iy : x \epsilon R_j,\ y = 0\} \qquad j = -6(1)6$$

$$B_k = \{x + iy : x = 0,\ y \epsilon R_k\} \qquad k = -6(1)6$$

$$C_{jk} = \{x + iy : x \epsilon R_j,\ y \epsilon R_k\} \qquad j, k = -6(1)6,$$

using a pseudorandom number generator [3]. One of the two arguments was used to generate I's and the other J's. For each argument, functions of order 0, 1, . . ., 10 were tested. It should be noted that the sets C_{jk} include complex arguments with real and imaginary parts of quite disparate magnitudes.

Summary statistics for the tests are:

Test on 60 Significant Bits

903	results correct to 60 bits*
1329	results in error by 1 multiple of last bit
967	results in error by 2 multiples of last bit
657	results in error by 3 multiples of last bit
404	results in error by 4 multiples of last bit
333	results in error by 5 multiples of last bit
247	results in error by 6 multiples of last bit
208	results in error by 7 multiples of last bit
1422	results in error by more than 7 multiples of last bit
6470	(Total)

Test on 60 Binary Places

320	results correct to 60 places**
479	results in error by 1 multiple of last place
279	results in error by 2 multiples of last place
160	results in error by 3 multiples of last place
82	results in error by 4 multiples of last place
59	results in error by 5 multiples of last place
45	results in error by 6 multiples of last place

*Excluding 202 results for pure real or imaginary argument, for which test and check values are both zero.

**Excluding 370 results for pure real or imaginary argument.

26 results in error by 7 multiples of last place
88 results in error by more than 7 multiples of last place

1538 (Total)

In the first test, the maximum error was $(6651)_8$. This occurred in the real part of $J_n(z)$ at $n=10$ and $z=(.1216\ldots)-i(.01928\ldots)$, for which $J_n(z)=-2^{-71}\cdot p-i\cdot 2^{-62}\cdot q$, with $\frac{1}{2}\leq p, q<1$. It is appropriate to assess this error relative to $|J_n(z)|$. This can be done fairly simply by dividing the bit error by 2^9, since 9 is the difference between the binary exponents of $\mathrm{Re}\, J_n(z)$ and $\mathrm{Im}\, J_n(z)$. This "relative error" is 7. On applying this technique to all the bit errors, the maximum relative error was found to be $(34)_8$. Allowing for the fact that one mantissa can be no more than twice the other, we see that the relative error is bounded by $(70)_8$, showing that BESLCI yielded values correct to 54 bits out of 60.

In the test on 60 binary places, the maximum error was found to be $(77)_8$, showing that at least 54 binary places are correct. (Indeed, the first 57 places were correct in 94% of the cases.)

The test on the two-term ascending series used only the argument set C_{jk} (see above), where j and k each take the values -16 and -100. The relative error was bounded by $(60)_8$, in comparison with $(70)_8$ for the test described above.

3. Testing of Error Return

The following arguments $z=x+iy$ were used:

$$\begin{array}{ll} 2^j+i\cdot 2^k & j,\ k=-9(3)12 \\ 2^j & j\ \ =-9(3)12 \\ i\cdot 2^k & k\ \ =-9(3)12; \end{array}$$

the values of NB are described in section 1. After the J's were calculated using $x+iy$ as the argument, the I's were calculated using $y+ix$. This showed that BESLCI gave results in accord (to 60 bits) with the equations

$$J_n(\bar{z})=\overline{J_n(z)}, \qquad I_n(z)=(-i)^n J_n(iz).$$

In the test on Bessel function values of order NCALC−1, the largest error was 5 multiples of the last bit; most were zero.

In the test on order $[|z|]$, the errors were larger, owing to the greater number of back-recursion steps. To calculate $J_{4096}(4096+i\cdot 2^{-9})$, for example, the subroutine call with NB = 5600 involved approximately 900 more steps than the call with NB = 4097. Bearing in mind that each step includes 2 multiplications and 3 additions for both the real and imaginary parts of the function value and also that the computer uses truncation in floating-point arithmetic rather than roundoff, we see that the error of $(2234)_8=1180$ in the last bit in the real part of $J_{4096}(4096+i\cdot 2^{-9})$ is to be expected. The imaginary part of $J_{4096}(4096+i\cdot 2^{-9})$ was about 2^{-15} times the size of the real part, and the bit error, $(72461407)_8$, was about 2^{14} times as big, which, again, is to be expected.

The errors mentioned in the preceding paragraphs were the worst cases. For all other arguments, the errors were within the expected tolerance, based on the number of recursion steps, and the relative sizes of x and y.

4. Summary and Conclusions

For $n=0(1)10$ and the values of z tested, BESLCI yielded $J_n(z)$ and $I_n(z)$ on the UNIVAC 1108 with either a relative error or an absolute error bounded by 10^{-16}. The latter assessment applied

when $n < |z|$ and the magnitude of the real or imaginary part of the Bessel function value being tested is less than $\frac{1}{2}$; the former applies in all other cases.

The error return feature was tested and found to be executed accurately in all cases. This test also confirmed the error bounds described above for values of z in the square $|\text{Re } z| < 64$ and $|\text{Im } z| < 64$.

For other values of n and z, similar accuracy may be expected since the tests we have used include large and small values of $|z|$ and $|z|/n$.

Calculations were performed on a UNIVAC 1108 under EXEC 2. All were carried out with double-precision mantissa of 60 bits, corresponding to just over 18 figures. Subroutine executions took approximately 0.5N milliseconds, where

$$N = \max\ (|z|, \text{NB}) + 10.$$

5. References and Note

[1] Algorithm BESLCI, see previous paper.

[2] Maximon, Leonard C., FORTRAN Program for Arbitrary, Precision Arithmetic, unpublished data.

[3] A linear congruential generator $X_{n+1} = aXn \pmod{2^{60}}$ with $Xo = a = 5^{13}$. The 60 bits generated were used as the mantissa, and the complement of the leftmost bit taken for the sign bit.

(Paper 77B3&4–388)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 3 & 4, July–December 1973

Triangles Generated by Powers of Triplets on the Unit Circle

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(June 7, 1973)

Let α, β, γ be three distinct complex numbers of modulus 1. It is shown that there is essentially one exception to the following statement: For some positive integer m, 0 is in the closed convex hull of $\alpha^m, \beta^m, \gamma^m$. The exception occurs for the normalized triple

$$1, e^{2\pi i/7}, e^{2k\pi i/7},$$

where $k=3$ or 5. This question was motivated by the problem of determining when a positive integer m and a nonzero $n\times 1$ vector x exist such that

$$x^*A^m x=0,$$

where A is a given matrix of $M_n(C)$.

Key words: Convex hull; unit circle; Weyl's Theorem.

In connection with determining for what $A\epsilon M_n(C)$ the equation

$$x^*A^m x=0$$

is solvable by some positive integer [1] m and some $n\times 1$ vector $x\neq 0$, the following question arose: How many distinct points $\alpha_1, \ldots, \alpha_l$ on the unit circle are in general required to insure that for some positive integer m, 0 lies in the convex hull of $\{\alpha_1^m, \ldots, \alpha_l^m\}$? We find that in general $l=4$ such points are required. However, our main result is that under appropriate normalization in the case $l=3$ there is exactly one exceptional set.

Throughout α, β, γ will denote three distinct points on the unit circle in the complex-plane. We shall denote the triangular solid generated by their mth powers by

$$T_m(\alpha, \beta, \gamma)=\mathrm{Co}\{\alpha^m, \beta^m, \gamma^m\}.$$

Our goal is to determine for which triples α, β, γ, there is a positive integer m such that

$$0\epsilon T_m(\alpha, \beta, \gamma).$$

For this purpose we shall identify two triples (α, β, γ), and $(\alpha', \beta', \gamma')$ if one may be obtained from the other via any combination of permutation, reflection, and simultaneous rotation. We shall also identify these two triples if

$$\{T_m(\alpha, \beta, \gamma)|m\epsilon I^+\}=\{T_m(\alpha', \beta', \gamma')|m\epsilon I^+\}.$$

AMS Subject Classification: 52-00, 15A24, 15-A63.

*The work of the first author was done while he was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

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Subsequent to this identification there is exactly one exception (α, β, γ) to the statement:

There is an $m \in I^+$ such that $0 \in T_m(\alpha, \beta, \gamma)$.

With respect to the identification mentioned above, we shall take our triples to be in normalized form:

$$\alpha = e^{2\pi i z_1},\ \beta = e^{2\pi i z_2},\ \gamma = 1$$

where $0 < z_1 < z_2 < 1$. Letting $\{r\}$ denote the fractional part of the real number r, we shall then say that the positive integer m is a solution to the normalized problem (*) if

$$\text{(i)}\ \{mz_{j_1}\} \leqslant 1/2$$

$$\text{(ii)}\ \{mz_{j_2}\} \geqslant 1/2,$$

and

$$\text{(iii)}\ \{mz_{j_2}\} - \{mz_{j_1}\} \leqslant 1/2,$$

where $(j_1, j_2) = (1, 2)$ or $(2, 1)$. It is clear that m is a solution to the normalized problem (*) if and only if $0 \in T_m(\alpha, \beta, \gamma)$.

Example. If $z_1 = \frac{1}{7}$ and $z_2 = \frac{3}{7}$ or $\frac{5}{7}$, then the system (*) is not solvable. Only the values $1 \leqslant m \leqslant 7$ need be considered and it is routine to check that for each of these values at least one of (i), (ii), or (iii) is not satisfied.

Our main result is:

THEOREM: *Let α, β, γ be distinct complex numbers such that $|\alpha| = |\beta| = |\gamma| = 1$. Then there is a positive integer* m *such that* $0 \in \mathrm{T_m}(\alpha, \beta, \gamma)$ *if and only if the normalized form of* $\{\alpha, \beta, \gamma\}$ *is not* $\{e^{\frac{2\pi j}{7}}, e^{\frac{2k\pi j}{7}}, 1\}$, $k = 3$ *or* 5.

PROOF: It suffices to consider the normalized problem (*). The necessity then follows from the given example. For the sufficiency we distinguish 5 possibilities (1) z_1 and z_2 are irrational and rationally independent; (2) z_1 and z_2 are irrational and rationally dependent; (3) exactly one of z_1 and z_2 is rational; (4) z_1 and z_2 are rational with distinct denominators in reduced form; (5) z_1 and z_2 are rational with the same denominator in reduced form.

For cases 1, 2, and 3 we shall employ a well-known theorem of Weyl:

LEMMA 1 (Weyl [2]): (a) If z is irrational, then the sequence $\{nz\}_{n=1}^{\infty}$ is uniformly distributed on the unit interval. (b) If z_1 and z_2 are rationally independent, then the ordered pairs $(\{nz_1\}, \{nz_2\})$, $n = 1, 2, \ldots$, are uniformly distributed on the unit square.

In case 1, the normalized problem (*) is easily solved because of Lemma 1, part (b). For case 2 we assume without loss of generality that the pair z_1, z_2 is of the form

$$bz,\ az + r$$

where z is irrational, $r = \frac{l_1}{l_2}$ is rational, and a, b, l_1, l_2 are integers with $a, b, l_2 > 0$. If $a = b$, this case may be transformed into case 3 by rotation. Thus we also assume without loss of generality that $a > b$. Suppose $m = m'l_2$, $m' > 0$ and integral, and let

$$c = \{mz\} = \{m'(l_2 z)\}.$$

Then by Lemma 1, part (a), we may obtain c arbitrarily close to any real number between 0 and 1 by choice of m'. Since $a > b > 0$, we may choose ϵ_1, ϵ_2 so that

$$\epsilon_1 > 0,\ \epsilon_2 > 0,\ \epsilon_1 + \epsilon_2 < \frac{1}{2}$$

and

$$\frac{b}{a} = \frac{1-2\epsilon_2}{1+2\epsilon_1}.$$

Next choose

$$0 < \epsilon < \min\left\{\frac{1}{2a} - \frac{\epsilon_1}{a}, \frac{\epsilon_2}{b}\right\}.$$

Then choose m' so that

$$0 \leq c - \left(\frac{1}{2a} + \frac{\epsilon_1}{a}\right) < \epsilon.$$

We then have

$$\{m(az+r)\} = \{ac\} = ac$$

$$\{m(bz)\} = \{bc\} = bc$$

and

$$\{m(az+r)\} - \{m(bz)\} = ac - bc.$$

Since

$$\frac{1}{2} + \epsilon_1 \leq ac < \frac{1}{2} + \epsilon_1 + \epsilon a$$

and

$$\frac{1}{2} - \epsilon_1 \leq bc < \frac{1}{2} - \epsilon_2 + \epsilon b,$$

it follows that

$$\frac{1}{2} \leq ac \leq 1$$

$$0 \leq bc \leq \frac{1}{2}$$

and

$$0 \leq ac - bc \leq \frac{1}{2}$$

so that (*) is solvable in case 2. In case 3, (*) may be solved using Lemma 1, part (a).

For the remaining cases we assume that in reduced form

$$z_1 = \frac{h}{n_1},\ z_2 = \frac{k}{n_2}.$$

Without loss of generality we may assume that h or k is 1. If either n_i is even, then $0 \epsilon T_{n_{i/2}}(\alpha, \beta, 1)$ and we are finished. Thus we may assume that n_1 and n_2 are odd. Suppose (case 4) that $n_1 \neq n_2$, and $n =$ g.c.d. (n_1, n_2). By the Chinese Remainder Theorem, the congruences

$$2hm \equiv n_1 - n \bmod n_1$$

and

$$2km \equiv n_2 + n \bmod n_2$$

have a solution m, which may be taken positive. It then follows that for such an m, (i), (ii), and (iii) are satisfied which completes the discussion of this case.

Finally we assume (case 5) that $n_1 = n_2 = n$ which is odd. Without loss of generality we take $h = 1$ and since z_2 is in reduced form we have g.c.d. $(k, n) = 1$.

LEMMA 2: If $n > 2$, $1 < k < n$ are integers, $z_1 = \frac{1}{n}$, $z_2 = \frac{k}{n}$, and $(k, n) = 1$, then the following are equivalent:

(1) the system (*) is solvable for $k = x$;
(2) the system (*) is solvable for $k = x'$, where $xx' \equiv 1 \pmod{n}$; and
(3) the system (*) is solvable for $k \equiv (1 - x) \pmod{n}$.

PROOF: The equivalence of (1) and (2) follows from the fact

$$\{T_m(e^{\frac{2\pi i}{n}}, e^{\frac{2\pi i x}{n}}, 1) \mid m \epsilon I^+\}$$

$$= \{T_m(e^{\frac{2\pi i}{n}}, e^{\frac{2\pi i x'}{n}}, 1) \mid m \epsilon I^+\}.$$

The equivalence of (1) and (3) follows from the fact that

$$e^{\frac{2\pi i}{n}}(e^{\frac{2\pi i}{n}}, e^{\frac{2\pi i x}{n}}, 1) = (1, e^{\frac{2\pi i(1-x)}{n}}, e^{\frac{2\pi i}{n}}).$$

Now if k is even, we may choose $m = \frac{n-1}{2}$ to satisfy (*) since $n \neq 1$ is odd. Thus we may assume because of Lemma 2 that k and k' are both odd, where $1 < k' < n$ is the unique solution to $kk' \equiv 1 \bmod n$. Because of Lemma 2, we may also assume that

$$1 < k \leqslant \frac{p+1}{2}.$$

However since $\left(\frac{p+1}{2}\right)' = 2$ which is even, we need only consider

$$1 < k \leqslant \frac{p-1}{2}.$$

We now wish to determine for which n (*) is solvable under our present assumptions.

Let $m \equiv k'\left(\frac{n+j}{2}\right) \bmod n$ where j *is odd* and $n < k'j < 2n$. Then since k' is odd, we have $\left\{\frac{m}{n}\right\} = \left\{\frac{1}{2} + \frac{k'j}{2n}\right\} = \left\{1 + \frac{k'j-n}{2n}\right\} = \left\{\frac{k'j-n}{2n}\right\} = \frac{k'j-n}{2n} < \frac{n}{2n} = \frac{1}{2}$ which means that (i) is satisfied.

If $1 < j < n$ also, then $\left\{\frac{mk}{n}\right\} = \left\{\frac{n+j}{2n}\right\} = \left\{\frac{1}{2} + \frac{j}{2n}\right\} = \frac{1}{2} + \frac{j}{2n} > \frac{1}{2}$ so that (ii) is satisfied.

Finally $\left\{\frac{mk}{n}\right\}-\left\{\frac{m}{n}\right\}=\frac{1}{2}+\frac{j}{n}-\frac{k'j-n}{2n}=1-\left(\frac{(k'-1)j}{2n}\right)$ which is less than $\frac{1}{2}$ *if* $(k'-1)j>n$.

We now have 4 requirements on j for (*) to be solvable. Since $k' \neq 1$ is odd they reduce to

$$j \text{ odd}$$

and

$$\frac{n}{k'-1}<j<\frac{2n}{k'}$$

Thus if the interval $\left(\frac{n}{k'-1}, \frac{2n}{k'}\right)$ is of length greater than or equal to 2, there will be a solution with j odd and integral. Thus we require

$$\frac{2n}{k'}-\frac{n}{k'-1} \geqslant 2$$

or

$$\frac{2(k'-1)n-nk'}{k'(k'-1)} \geqslant 2$$

or

$$n \geqslant 2\,\frac{k'(k'-1)}{(k'-2)}\,.$$

As a function of k', $2\,\frac{k'(k'-1)}{(k'-2)}$ is increasing for $k'>2+\sqrt{2}$. If $k'=\frac{n-3}{2}$, our requirement becomes

$$n \geqslant \frac{(n-3)(n-5)}{(n-7)}$$

or

$$n \geqslant 15.$$

It remains to check the cases in which $k'=3$ or $\frac{n-1}{2}$ for general odd n and the cases $n=3$, 5, 7, 9, 11, 13. By straightforward computation, the latter yield that (*) is solvable in all cases except that mentioned in the example. In the former case we have that

$$1-\left(\frac{n-1}{2}\right)^{r} \equiv 3 \bmod n$$

since n is odd. Thus by Lemma 2 it suffices to check $k=3$.

In case $k=3$, it is easily checked that $m=\left[\frac{n}{6}\right]+1$, where $[\cdot]$ denotes the greatest integer function, satisfies (*) for $n \geqslant 12$. The remaining case $n<12$ have already been checked so that the proof is complete.

COROLLARY: *If* $l \geqslant 4$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_l$ *are distinct complex numbers of absolute value* 1, *then there is a positive integer* m *such that*

$$0 \in \mathrm{Co}\{\alpha_1^m, \ldots, \alpha_l^m\}.$$

PROOF: Because of the Theorem, it suffices to check the set of four points $e^{\frac{2\pi i}{7}}, e^{\frac{6\pi i}{7}}, e^{\frac{10\pi i}{7}}, 1$. Since 0 is actually in their convex hull, the result is confirmed.

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(Paper 77B3&4–389)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 77B, Nos. 3 & 4, July–December 1973

Tables and Graphs of the Stable Probability Density Functions *

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(May 4, 1973)

Four-decimal-place tables are presented of the probability density function $p(x; \alpha, \beta)$ of the stable distribution for $\alpha = 0.25(.25)2.00$, $\beta = -1.00(.25)1.00$, and nonnegative x in steps varying by factors of 10 from 0.001 to 100 such that interpolation is possible, the tabulation being terminated where $p(x; \alpha, \beta)$ falls to 0.0001. The largest such value of x is 338, for $\alpha = 0.25$, $\beta = -1.00$. Graphs of $p(x; \alpha, \beta)$ are also provided for essentially the above values of α and β. The methods of calculation (from the characteristic function), checking, and interpolation with respect to x, β, and (to some extent) α are described. The most important properties of stable distributions are summarized. Some applications are cited. A selected bibliography with 91 items is included.

Key words: Accuracy; approximations; asymptotic expansion; Cauchy distribution; characteristic function; closed forms; contour; convergence; curves; distribution function; error; Fourier transform; infinite series; interpolation; limit distribution; normal distribution; polynomials; probability density function; quadrature; stable distribution; sums of independent random variables; tables; truncation.

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AMS Subject Classification: 60–00.

*The paper was begun when the second author was employed by the National Bureau of Standards, continued while he worked for the Environmental Science Services Administration (now National Oceanic and Atmospheric Administration), and completed under the Office of Telecommunications, all in Boulder, Colorado.

1. Introduction

The stable distributions have provided a fascinating and fruitful area of research for probability theorists and models of value in physics, astronomy, economics, and communications theory. The *symmetric* stable distributions were introduced by Cauchy in 1853 [4],[1] and one of them is well known by his name. The general class was studied systematically by Paul Lévy in the early 1920's [18a, 19]. If two independent real random variables with the same shape or *type* of distribution are combined linearly and the distribution of the resulting random variable has the same shape, the common distribution (or its type, more precisely) is said to be *stable*. The normal, or Gaussian, distribution is the best-known example.

The inspiration for systematic research on stable distributions was the desire to generalize the celebrated Central Limit Theorem, which states that under fairly general conditions, the distribution function (df) of the sum of n independent random variables approaches (when standardized to constant median and scale) the normal df as n becomes infinite. It was known that the mean of n independent observations from a Cauchy distribution has the same distribution as a single observation, which is thus also the limit distribution as $n \to \infty$ and not covered by the Central Limit Theorem. It can be shown that all limit distributions of sums of independent random variables must be stable [Lévy, 19; Gnedenko and Kolmogorov, 11].

The restrictive condition of stability enabled A. Ya. Khintchine and Lévy in 1936 [see 11, p. 164] to derive the general form for the characteristic function (*cf*, the Fourier transform of the probability density function (pdf)) of a stable distribution. However, the pdf itself is not known in closed form except for the normal and Cauchy types and one other type (see sec. 2.13).

The considerable interest in stable distributions makes a general set of tables of the densities and distribution functions highly desirable. A table of pdf's for characteristic exponent $\alpha = 1.1(.1)1.9$, 1.95, 1.99 was given by Mandelbrot and Zarnfaller in 1961 [88]. Fama and Roll [85] published tables of the df's and fractiles of standardized symmetric stable distributions for characteristic exponent $\alpha = 1.0(.1)1.9(.05)2.0$ (as well as formulas and tables for estimating the parameters). Here we present four-decimal-place tables (sec. 12) of the standardized pdf $p(x; \alpha, \beta)$ for $\alpha = 0.25(.25)2.00$, $\beta = -1.00(.25)1.00$, and nonnegative x in steps varying by factors of 10 from 0.001 to 100 such that interpolation is possible. The tabulation is terminated where $p(x; \alpha, \beta)$ first falls to 0.0001, correct to the fourth decimal place. The largest such value of x is 338, for $\alpha = 0.25$, $\beta = -1.00$. Graphs of $p(x; \alpha, \beta)$ are also provided for the above values of α and β (sec. 13). The methods of calculation, checking, and interpolation are described in detail (secs. 4–8). Probabilities may be calculated from the tables by Simpson's Rule (sec. 10).

The theory and properties of stable distributions have been systematically presented by Lukacs [23, 24], Gnedenko and Kolmogorov [11], and Feller [44], but a brief statement, without proofs, of the most important properties, taken from these sources, especially [24], is given in section 2 for convenience. Some applications are noted in section 3. An extensive selected bibliography is included (sec. 11).

2. Definition and Properties of Stable Distributions

2.1. *Definition.*

The (cumulative) *distribution function* (df), $P(x)$, of a real random variable X is the probability that X is less than or equal to the real number x:

$$P(x) \equiv \text{Prob } [X \leq x], \qquad -\infty < x < +\infty.$$

[1] Figures in brackets indicate the literature references on pages 161–164.

(The definition of a random variable is omitted, being outside the scope of this paper.)

2.2. *Definition.*

The *probability density function* (*pdf*), of X is (when it exists or, equivalently, when $P(x)$ is absolutely continuous) the derivative of the *df*:

$$p(x) \equiv P'(x) \equiv dP/dx, \qquad -\infty < x < +\infty.$$

2.3. *Definition.*

The *characteristic function* (*cf*) of X is the Fourier-Stieltjes transform of the *df*:

$$\varphi(t) \equiv \int_{-\infty}^{\infty} e^{itx} dP(x), \qquad -\infty < t < +\infty,$$

$$= \int_{-\infty}^{\infty} e^{itx} p(x) dx$$

when $p(x)$ exists.

2.4. The *df*, *pdf* (when it exists), and *cf* are alternative and equivalent ways of describing a probability distribution, and it is at times convenient and sufficient to say "distribution" rather than *df*, *pdf*, or *cf*. When the *pdf* $p(x)$ exists,

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \varphi(t) dt.$$

2.5. *Definition.*

If X_1 and X_2 are two independent random variables with *df*'s $P_1(x)$ and $P_2(x)$ and *cf*'s $\varphi_1(t)$ and $\varphi_2(t)$, then the *df* $P(x)$ of the sum $X \equiv X_1 + X_2$ is given by the convolution of $P_1(x)$ and $P_2(x)$:

$$P(x) = P_1(x) * P_2(x) \equiv \int_{-\infty}^{\infty} P_1(x-y) dP_2(y).$$

The corresponding *cf* is $\varphi(t) = \varphi_1(t)\varphi_2(t)$.

2.6. *Definition.*

Two *df*'s $P(x)$ and $Q(x)$ are *of the same type* if there exist a postive c and real d such that the relation $Q(x) = P((x-d)/c)$ holds for all x.

2.7. *Definition.*

A *df* $P(x)$ (or any *df* of the same type as $P(x)$) is *stable* if to every $c_1 > 0$, $c_2 > 0$ and real d_1, d_2 there correspond a positive c and real d such that

$$P\left(\frac{x-d_1}{c_1}\right) * P\left(\frac{x-d_2}{c_2}\right) \equiv P\left(\frac{x-d}{c}\right)$$

for all x. An equivalent condition for $P(x)$ to be stable is the following: If X_1 and X_2 are independent random variables with the df $P(x)$, then the df of $c_1X_1 + c_2X_2$ is of the same type as $P(x)$ for every $c_1 > 0$, $c_2 > 0$. Stability is really a property of the *type* of distribution rather than of any single distribution, but we may attribute it conveniently to a *df*, a *pdf*, or a *cf* also, just as we speak of the normal distribution when we mean the normal type.

2.8. A cf $\varphi(t)$ is stable if and only if it has the form

$$\varphi(t) = \exp\{idt - |ct|^\alpha[1 + i\beta(t/|t|)\omega(|t|, \alpha)]\}$$

where $-\infty < d < +\infty$, $c \geq 0$, $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, and

$$\omega(|t|, \alpha) = \begin{cases} \tan(\pi\alpha/2) & \text{for } \alpha \neq 1 \\ (2/\pi)\log|t| & \text{for } \alpha = 1. \end{cases}$$

(Here c is the scale parameter as in Fama and Roll [85] but not as in Lukacs [23, 24]. If the c of Lukacs is denoted by c', then $c' = c^\alpha$.) The constant α is called the *characteristic exponent*. (Unfortunately the above definitions of ω produce the heavier tail of the *pdf* in the positive x direction for $\alpha = 1$ and in the negative x direction for $\alpha \neq 1$. Thus there is an apparent inconsistency in the tables and graphs.)

2.9. *Degenerate distribution.*

If $c = 0$, the stable cf reduces to that of the degenerate (improper) distribution with df

$$P(x) = \begin{cases} 0, & x < d \\ 1, & x \geq d. \end{cases}$$

2.10. *Normal distribution.*

If $\alpha = 2$, the stable cf reduces to that of the normal distribution with mean d and variance $2c^2$ (independent of the value of β).

2.11. *Cauchy distribution.*

If $\alpha = 1$ and $\beta = 0$, the stable cf reduces to that of the Cauchy distribution with median d, semi-interquartile range c, and *pdf*

$$\frac{1}{\pi} \cdot \frac{c}{c^2 + (x-d)^2}.$$

For $c = 1$ and $d = 0$ this reduces to the Student t *pdf* with 1 degree of freedom.

2.12. The cf of the standardized variable $(X - d)/c$ for $c > 0$ is of the form in section 2.8 with $d = 0$ and $c = 1$. Hereafter (except in secs. 2.24 and 2.28) we therefore take $d = 0$ and $c = 1$ and write the *pdf* of the standardized variable, by section 2.4, as

$$p(x; \alpha, \beta) = \frac{1}{\pi}\int_0^\infty e^{-t^\alpha}\cos[xt + \beta t^\alpha\omega(t, \alpha)]dt.$$

2.13. $\alpha=\frac{1}{2}, \beta=-1.$

Then the stable *cf* corresponds to the *pdf*

$$p(x; \tfrac{1}{2}, -1) = \begin{cases} (2\pi)^{-1/2} x^{-3/2} e^{-1/(2x)} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

This is the *pdf* of the reciprocal of a chi-square variable with 1 degree of freedom. It is also of Type V in the Pearson system of frequency curves. No stable distributions with *pdf*'s that are elementary functions other than the four above are known.

2.14. *Higher transcendental functions.*

The stable densities for several rational values of α can be represented in terms of higher transcendental functions as follows:

α	β	*Principal function involved*
1/3	1	Macdonald function (modified Bessel function of third kind) $K_{1/3}(x)$ [38]
2/3	0	Whittaker function $W_{1/2,\,1/6}(x)$ [38; 39; 69, p. 505]
2/3	1	
3/2	1	
1/2	arbitrary	$w(z)$ as shown in (7.1)−(7.3) below

2.15. $p(-x; \alpha, -\beta) = p(x; \alpha, \beta).$

Hence we tabulate $p(x; \alpha, \beta)$ only for nonnegative values of x.

2.16. *Continuity.*

All stable *df*'s are absolutely continuous. All stable *pdf*'s are continuous.

2.17. *Regularity.*

The stable *pdf*'s with $\alpha \geqq 1$ are regular (analytic) for all real x and are entire functions except for the Cauchy distributions. The stable *pdf*'s with $\alpha < 1$ have the form

$$p(x; \alpha, \beta) = \begin{cases} (1/x)\Phi_1(x^{-\alpha}) & \text{for } x > 0 \\ (1/|x|)\Phi_2(|x|^{-\alpha}) & \text{for } x < 0 \end{cases}$$

where $\Phi_1(z)$ and $\Phi_2(z)$ are entire functions.

2.18. *Derivatives.*

Derivatives of the stable *pdf*'s with respect to x of all orders exist for all real x. They are bounded according to

$$\left| \frac{\partial^n p(x; \alpha, \beta)}{\partial x^n} \right| \leq \frac{1}{\pi\alpha} \Gamma\left(\frac{n+1}{\alpha}\right).$$

2.19. *Unimodality.*

All stable *pdf*'s are unimodal.

2.20. *Symmetry.*

A stable *pdf* is symmetrical if and only if $\beta=0$.

2.21. *Convergent infinite series.*

The stable *pdf*'s have convergent infinite series representations. Let $x>0$, $\eta=\beta\tan(\pi\alpha/2)$, $r=(1+\eta^2)^{-1/(2\alpha)}$, and $\gamma=-(2/\pi)\arctan\eta$. Then the *pdf* of the standardized variable evaluated at the abscissa rx is, for $0<\alpha<1$,

$$p(rx;\,\alpha,\beta)=\frac{1}{\pi rx}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(\alpha k+1)x^{-\alpha k}\sin\left[\frac{k\pi}{2}(\alpha+\gamma)\right],$$

and for $1<\alpha\leq 2$,

$$p(rx;\,\alpha,\beta)=\frac{1}{\pi rx}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma\left(\frac{k}{\alpha}+1\right)x^{k}\sin\left[\frac{k\pi}{2\alpha}(\alpha+\gamma)\right].$$

A series in powers of x exists for $\alpha=1$, but the coefficients are expressed only as complicated integrals.

2.22. *Asymptotic series.*

Asymptotic series are available for x in the neighborhood of 0, $+\infty$, or $-\infty$ and exhaustive ranges of values of α and β. In particular the first n terms of the first series in section 2.21 is an asymptotic approximation for $x\to+\infty$ and $1<\alpha<2$. Likewise the first n terms of the second series in section 2.21 is an asymptotic approximation for $x\downarrow 0$ and $0<\alpha<1$. (Note the interchange in the ranges of α from those for convergence in sec. 2.21.)

2.23. $x=0$, $\alpha\neq 1$.

By changes of variable in section 2.12 with $x=0$ and $\alpha\neq 1$ we obtain gamma functions and consequently

$$p(0;\,\alpha,\beta)=\frac{1}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left(\cos\frac{\pi\gamma}{2}\right)^{1/\alpha}\cos\frac{\pi\gamma}{2\alpha},$$

with γ as in section 2.21.
In particular

$$p(0;\,\alpha,0)=\frac{1}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right);$$

$$p(0;\,\alpha,\pm 1)=0,\ \alpha<1;$$

$$p\left(0;\,\frac{1}{2},\beta\right)=\frac{2}{\pi}\frac{1-\beta^2}{(1+\beta^2)^2};$$

$$p\left(0;\,\frac{1}{3},\beta\right)=\frac{3!}{\pi}\frac{1-\beta^2}{(1+\beta^2/3)^3};$$

$$p\left(0; \frac{1}{4}, \beta\right) = \frac{4!}{\pi} \frac{1-6\eta^2+\eta^4}{(1+\eta^2)^4}, \qquad \eta = \beta(\sqrt{2}-1);$$

$$p\left(0; \frac{2}{3}, \beta\right) = \frac{3}{4\sqrt{2\pi}} \rho^{-3}(2-\rho)(1+\rho)^{1/2}, \qquad \rho = (1+3\beta^2)^{1/2}.$$

2.24. *Interrelation for α and $\alpha^* = 1/\alpha$.*

It is convenient to consider a stable *pdf* of the nonstandardized variable $Y = X/c$, where $c = [\cos(\pi\gamma/2)]^{1/\alpha}$ with γ as in section 2.21. Let $1 < \alpha \leq 2$, $\alpha^* = 1/\alpha$, $\gamma^* = (\gamma+\alpha-1)/\alpha$, and $p_{\alpha\gamma}(y)$ be the *pdf* of Y. Then for $y > 0$

$$p_{\alpha^*\gamma^*}(y) = y^{-\alpha^*-1} p_{\alpha\gamma}(y^{-\alpha^*}).$$

2.25. *Zero values of pdf's.*

If $0 < \alpha < 1$ and $\gamma = -\alpha$, then it follows from the first formula in section 2.21 that

$$p(x; \alpha, 1) = 0 \text{ for } x > 0.$$

From section 2.15

$$p(x; \alpha, -1) = 0 \text{ for } x < 0 \text{ and } 0 < \alpha < 1.$$

2.26. *Infinite divisibility.*

A stable *cf* is infinitely divisible; i.e., for every positive integer n it can be expressed as the nth power of some *cf*. Equivalently we can say that for every positive integer n a stable *df* can be expressed as the n-fold convolution of some *df*.

2.27. *Moments.*

The absolute moment of order δ of a *df* $P(x)$ is defined as

$$\int_{-\infty}^{\infty} |x|^{\delta}\, dP(x) \text{ or } \int_{-\infty}^{\infty} |x|^{\delta}\, p(x)\, dx$$

when $p(x)$ exists. Every stable *df* with characteristic exponent α with $0 < \alpha < 2$ has finite absolute moments of all orders δ with $0 < \delta < \alpha$, whereas all absolute moments of order $\delta \geq \alpha$ are infinite. Thus the normal *df's* are the only stable *df's* with finite variance. The mean exists for $1 < \alpha \leq 2$, but neither the mean nor the variance exists for $0 < \alpha \leq 1$.

2.28. *Location and scale parameters of a linear combination.*

If $X_1, X_2, \ldots, X_n$ are independent random variables with stable distributions of the same type (and therefore with the same α and β), location parameters $d_1, d_2, \ldots, d_n$, and scale parameters $c_1, c_2, \ldots, c_n$ respectively, then the linear combination

$$Y = \sum_{i=1}^{n} a_i X_i, \qquad a_i \geq 0,$$

has, from sections 2.3, 2.5, and 2.8, the location parameter

$$d = \sum_{i=1}^{n} a_i d_i$$

and scale parameter c given by

$$c^\alpha = \sum_{i=1}^{n} a_i^\alpha c_i^\alpha.$$

The last equation is familiar for $\alpha = 2$, the normal distribution, but is not so well known for other α. In particular, if all $d_i = d_1$, $c_i = c_1$, and $a_i = 1/n$, so that Y is the sample mean $\bar{X}$, then

$$c = \frac{c_1}{\sqrt{n}}, \qquad \alpha = 2 \text{ (normal)}$$

$$c = c_1, \qquad \alpha = 1 \text{ (Cauchy if } \beta = 0)$$

$$c = nc_1, \qquad \alpha = \frac{1}{2} \text{ (Pearson Type V if } \beta = -1)$$

Thus the scale parameter (as well as the entire distribution) for $\alpha = 1$ is the same for $\bar{X}$ as for X_1, but the scale parameter of $\bar{X}$ for $\alpha = \frac{1}{2}$ (and all other $\alpha < 1$) increases without limit as $n \to \infty$.

2.29. *Importance in limit distributions.*

Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be independent and identically distributed random variables. Let

$$U_n = \frac{1}{B_n} \sum_{i=1}^{n} Z_i - A_n, \qquad n = 1, 2, \ldots,$$

where $A_1, A_2, \ldots$ and $B_1, B_2, \ldots, B_i > 0$, are sequences of constants. A well-known Central Limit Theorem states that if the Z_i have a finite variance then the sequence of *df*'s of the U_n converges to the standardized normal *df* for suitable choices of the A_i and B_i. The normal *df* is not the only possible limit *df* for the U_n in general, however: In order that a *df* be the limit *df* of a sequence of *df*'s of U_n it is necessary and sufficient that it be a stable *df*.

3. Applications of Stable Distributions

Beyond the pervasive occurrence of the normal distribution in practice as well as in theory, the primary interest in the stable distributions arises in probability theory as a result of their remarkable theoretical properties. However, they have also been applied in a wide variety of scientific fields, as indicated by the substantial but quite incomplete appended bibliography on applications. Feller's Volume II [44] discusses applications to the gravitational field of stars (Holtsmark distribution, $\alpha = 3/2$; cf. Holtsmark, [48]), first-passage times in Brownian motion ($\alpha = 1$), diffusion theory, and economics.

Benoit Mandelbrot has discussed many applications in economics, including the distributions of income and of speculative prices and relating stable distributions to the well-known Pareto distributions [53–59, 61, 63]. He has also found stable distributions in biology [51, 62], psychology

[52], and electrical engineering [60]. The occurrence in economics or business is also discussed by Press [66], Roll [67], Teichmoeller [68], Granger and Orr [47], and Fielitz and Smith [45].

The interest in economics led to the tabulation of the *cdf*'s and fractiles of the symmetric stable distributions for $\alpha = 1.0\ (.1)\ 2.0$ by Fama and Roll [85] and to estimation of parameters and tests of hypotheses by them [86] and by Press [28]. A manuscript including the *cdf*'s for certain asymmetric stable distributions ($\beta = -1\ (.25)0$) for $\alpha = 1.1\ (.2)\ 1.9, 2.0$, as well as graphs of the corresponding *pdf*'s for $\alpha = 1.1\ (.4)\ 1.9$, has just been sent to us by M. J. Cross [84].

The infinite divisibility of the stable distributions accounts for their occurrence in the theory of stochastic processes with stationary independent increments; see Feller [44, Vol. II], Kolmogorov and Sevast'yanov [50], Kac [14], Berman [41], Getoor [10]; Greenwood [12], Kesten [15, 49], and Lukacs [25].

Other occurrences in electrical engineering are indicated by Dobrushin [42] and by Ovsevič and Yaglom [65].

4. Approximation of the Stable Density $p(x; \alpha, \beta)$

The present tables of the stable *pdf* $p(x; \alpha, \beta)$ were calculated primarily from four approximations:

p_1, the first N terms of eq (4.1) below;
p_2, (4.12);
p_3, Section 4.3;
p_4, Section 7.

The values of α and β for which each of these is used are shown in table 1.

TABLE 1. *Approximations for the stable density* $p(x; \alpha, \beta)$

β \ α	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
−1.00	p_2[b]	$1/\chi_1^2$	p_1, p_4[a]	p_1	p_2	p_2[b, c]	p_1	Normal
−.75		p_2[b]				p_1		
−.50								
−.25			p_1, p_4	p_1	p_2			
.00	(c)	(c)	$p_2 (x \neq 0)$[c]	Cauchy	$p_1 (x \neq 0)$[c]	$(x \neq 0)$[c]	$(x \neq 0)$[c]	
.25			p_2	p_3	p_1			
.50			p_2					
.75	p_2[b]	p_2[b]	p_2					
1.00	0	0	0	p_3	p_1	p_1	p_1	Normal

[a] p_1 for $0 \leq x \leq x_N$, p_4 for $x_N \leq x \leq x_T$
[b] $0 < x_0 \leq x \leq x_T$
[c] See section 2.23

In all cases the *pdf* is computed over the interval $0 \leq x \leq x_T$, where x_T is such that $p(x_T; \alpha, \beta) = 0.00010$ and $p(x_{T-1}; \alpha, \beta) = 0.00011$ with $x_T = x_{T-1} + 0.1$. However, the tabulated *pdf* values are rounded to 4 decimal places so that several values of x usually exist for which $p(x; \alpha, \beta) = 0.0001$. For $\alpha = 0.75$, $\beta < 0$, and x large, say $x \geq x_N$, p_1 converges too slowly and p_4 is used for $x_N \leq x \leq x_T$. The approximation p_2 converges slowly for x near 0; hence, it is used only for $x \geq x_0 = 10^{-n}, n = 1, 2, 3$ with $n = 3$ only where it is essential ($\alpha = 0.25, 0.50$).

In sections 4.1 − 4.3 we derive the approximations from the integral formula for $p(x; \alpha, \beta)$ in section 2.12. We assume $x \geq 0$ throughout the remaining sections.

4.1. $p_1(x;\alpha,\beta)$.

To obtain $p_1(x;\alpha,\beta)$ we remove the branch point $t=0$ by setting $\alpha=p/q\neq 1$ (p, q positive integers) and $t=u^q$. Now consider the infinite series

$$p(x;\alpha,\beta)=\frac{q}{\pi}\sum_{k=0}^{\infty}\int_{u_k}^{u_{k+1}} e^{-u^p}u^{q-1}\cos f_1(u;\alpha,\beta,x)\,du \qquad (u_0=0) \tag{4.1}$$

such that u_k ($k=0,1,2,\ldots$) is the largest nonnegative real root of the function

$$|f_1(u;\alpha,\beta,x)|-k\pi\equiv\left|xu^q+\left(\beta\tan\frac{\pi\alpha}{2}\right)u^p\right|-k\pi. \tag{4.2}$$

To prove that the expansion (4.1) is an alternating series consider the proposition $u_{k+1}-u_k\to 0$ as $k\to\infty$. From (4.2) the difference equation for two successive extrema u_k and u_{k+1} yields

$$x\,(u_{k+1}^q-u_k^q)+\eta\,(u_{k+1}^p-u_k^p)=\pm\pi \tag{4.3}$$

where $\eta=\beta\tan(\pi\alpha/2)$. Equation (4.3) can be written as

$$u_{k+1}-u_k=\frac{\pm\pi}{xg_{q-1}(u_k,u_{k+1})+\eta g_{p-1}(u_k,u_{k+1})}$$

where $g_{n-1}(u_k,u_{k+1})=u_{k+1}^{n-1}+u_{k+1}^{n-2}u_k+\ldots+u_k^{n-1}$. Since $u_k\to\infty$ as $k\to\infty$, it follows that $u_{k+1}-u_k\to 0$.

We now show that for k sufficiently large

$$\int_{u_k}^{u'} e^{-u^p}u^{q-1}|\cos f_1(u;\alpha,\beta,x)|\,du\geq e^{-u'^p}u'^{q-1}\int_{u_k}^{u'}|\cos f_1(u;\alpha,\beta,x)|\,du$$

$$\geq e^{-u'^p}u'^{q-1}\int_{u'}^{u_{k+1}}|\cos f_1(u;\alpha,\beta,x)|\,du$$

$$\geq\int_{u'}^{u_{k+1}} e^{-u^p}u^{q-1}|\cos f_1(u;\alpha,\beta,x)|\,du, \tag{4.4}$$

where $\cos f_1(u_k;\alpha,\beta,x)=\pm 1$, $\cos f_1(u';\alpha,\beta,x)=0$, $\cos f_1(u_{k+1};\alpha,\beta,x)=\mp 1$, and $u_k<u'<u_{k+1}$. To demonstrate the middle inequality, let $v=xu^q+\eta u^p$. Then

$$\int_{u_k}^{u'}|\cos(xu^q+\eta u^p)|\,du=\int_{v_{\max}}^{v_0}|\cos v|\,\frac{dv}{xqu^{q-1}+p\eta u^{p-1}},$$

and, since $v_{\min}-v_0=v_{\max}-v_0=\pi/2$, the inequality

$$\int_{v_0}^{v_{\min}} |\cos v| \frac{dv}{xqu^{q-1}+\eta pu^{p-1}} \leq \int_{v_{\max}}^{v_0} \frac{|\cos v|}{xqu^{q-1}+\eta pu^{p-1}} dv$$

is true. Thus (4.4) is established. Since both the integrand and the interval $u_{k+1}-u_k$ approach zero as $k \to \infty$, so also does the kth term of the series (4.1).

Similarly when $\alpha=1$ we have the infinite series

$$p(x; 1, \beta)=\frac{1}{\pi}\sum_{k=0}^{\infty}\int_{t_k}^{t_{k+1}} e^{-t}\cos f_2(t; \beta, x)dt \qquad (t_0=0) \tag{4.5}$$

such that t_k is the largest nonnegative real root of the equation

$$|f_2(t; \beta, x)-k\pi|=\left|xt+\frac{2\beta}{\pi}t \ln t\right|-k\pi \qquad (k=0, 1, \ldots). \tag{4.6}$$

It is easy to show that $t_{k+1}-t_k \to 0$ as $k \to \infty$ by substituting the expansion

$$\ln t_{k+1}=\ln [t_k+t_{k+1}-t_k]$$

$$=\ln t_k+2\left[\frac{t_{k+1}-t_k}{t_{k+1}+t_k}+\frac{1}{3}\left(\frac{t_{k+1}-t_k}{t_{k+1}+t_k}\right)^3+\ldots\right]$$

into the difference equation

$$x(t_{k+1}-t_k)+\frac{2\beta}{\pi}(t_{k+1}\ln t_{k+1}-t_k \ln t_k)=\pm\pi.$$

Hence we have

$$t_{k+1}-t_k=\frac{\pm\pi}{x+\frac{2\beta}{\pi}\left[\ln t_k+2\frac{t_{k+1}}{t_{k+1}+t_k}+O\left(\left(\frac{t_{k+1}-t_k}{t_k}\right)^2\right]\right.}$$

Since $t_k \to \infty$ as $k \to \infty$, it follows that $t_{k+1}-t_k \to 0$.

The fact that (4.5) is an alternating series follows in the same way as in establishing (4.4).

The approximation $p_1(x; \alpha, \beta)$ is the truncation of (4.1) and (4.5) to N terms with the Nth term the first term not greater than 10^{-5} in magnitude.

The zeros u_k and t_k are computed via the Newton-Raphson (NR) and False Position (FP) root-finding procedures. The NR procedure is used in the following cases, for which the necessary sign conditions certainly hold:

$$\left.\begin{aligned}&\alpha=\frac{3}{2},\ -1<\beta<0\\&\alpha=\frac{7}{4},\ \beta<0\end{aligned}\right\}\ 0<x_0<x\leq x_T.$$

In the remaining cases the FP procedure is applied.

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Determining the initial values, say $u_{k,0}$ and $t_{k,0}$, for the NR and FP procedures in the presence of arbitrary fixed (x, α, β) requires a little care. For the NR procedure the following initial values were used:

$$u_0(x_i) = 0, u_{k,0}(x_0) = 1 (k = 1, 2, \ldots), \qquad u_{1,0}(x_i) = u_1(x_{i-1}),$$
$$u_{k,0}(x_i) = u_{k-1}(x_i) \ (k = 2, 3, \ldots), \qquad (i = 0, 1, \ldots) \tag{4.7}$$

where $u_k\ (x_i)$ is the root of eq (4.2) with $x = x_i$.

A table of initial values for the FP procedure is available on request.

We denote the kth integral of (4.1) (i.e., not including the factor q/π) by T_k, normalize the interval of integration to $[-1, 1]$, and approximate T_k by 48-point Legendre-Gaussian quadrature. Hence, the quadrature representation becomes

$$T_k \doteq A_k \sum_{j=1}^{48} H_j w_{jk}^{q-1} e^{-w_{jk}^p} \cos\ (x w_{jk}^q + \eta w_{jk}^p), \tag{4.8}$$

$$w_{jk} = A_k v_j + B_k, A_k = \frac{u_{k+1} - u_k}{2}, \text{ and } B_k = \frac{u_{k+1} + u_k}{2},$$

where H_j and v_j are Gaussian weights and abscissas respectively [79, p. 30].

The approximation for (4.5) parallels the approximation for (4.1); hence we omit that formulation.

Although the approximation $p_1(x; \alpha, \beta)$ may be used for any given (α, β, x), experience showed that the other approximations below required less computer time for the desired accuracy over certain ranges of α, β, x as shown in table 1.

4.2 $p_2(x; \alpha, \beta)$

Consider the contour integral

$$\frac{1}{\pi} \int_C \exp\left[-w^\alpha + i(xw + \eta w^\alpha)\right] dw \qquad (\alpha \neq 1) \tag{4.9}$$

taken on the path in figure 1.

Along the t axis (4.9) reduces to $p(x; \alpha, \beta)$ in the limit as $\epsilon \to 0$ and $R \to \infty$. The integral along C_1, say I_1, is bounded by

$$|I_1| \leq R \int_0^{\pi/2} \exp\left[-R^\alpha(\cos \alpha\theta + \eta \sin \alpha\theta) - xR \sin \theta\right] d\theta. \tag{4.10}$$

Hence, $I_1 \to 0$ as $R \to \infty$. The contribution on C_2 goes to zero as the branch point is approached, and consequently, equating real parts yields

$$p(x; \alpha, \beta) = \frac{1}{\pi} \int_0^\infty \exp\ (-xt - D_1 t^\alpha) \sin D_2 t^\alpha dt, \tag{4.11}$$

$$D_1 = \cos \frac{\alpha\pi}{2} + \beta \tan \frac{\pi\alpha}{2} \sin \frac{\pi\alpha}{2}, D_2 = (1 - \beta) \sin \frac{\pi\alpha}{2},$$

where $D_1 \geq 0$ is sufficient for the integral to converge ($D_1 > 0$ for $x = 0$).

Employing the transformation $t^{1/q}=u$ in (4.11), applying 48-point Legendre-Gaussian quadrature between each extremum pair of the function $\sin D_2u^p$ and normalizing the jth interval to $[-1, 1]$ yields the approximation

$$p_2(x;\alpha,\beta)=\frac{q}{\pi}\left(\frac{\pi}{2D_2}\right)^{q/p}\sum_{k=0}^{N}C_{1k}\left\{\sum_{j=0}^{48}H_j z_{jk}^{q-1}\cdot\exp\left\{-\left[x\left(\frac{\pi}{2D_2}\right)^{q/p}z_{jk}^{q}+\frac{D_1\pi}{2D_2}z_{jk}^{p}\right]\right\}\sin\frac{\pi}{2}z_{jk}^{p}\right. \tag{4.12}$$

where

$$z_{jk}=C_{1k}v_j+C_{2k},$$

$$C_{1k}=\tfrac{1}{2}[(2k+1)^{1/p}-(2k-1)^{1/p}],$$

$$C_{2k}=\tfrac{1}{2}[(2k+1)^{1/p}+(2k-1)^{1/p}], \qquad k=1, 2, \ldots, N,$$

$$C_{10}=C_{20}=\tfrac{1}{2},$$

H_j and v_j are Gaussian weights and abscissas respectively, and N is determined so that the Nth term is the first term not greater than 10^{-5} in magnitude.

4.3 $p_3(x; 1, \beta)$.

Using a different contour integration, Skorohod [32, p. 161] obtained a useful alternative integral formula for $\alpha=1$:

$$p(x;1,\beta)=\frac{1}{\pi}\int_0^\infty \exp\left[-xu-(2/\pi)\beta u\ln u\right]\sin\left[(1+\beta)u\right]du. \tag{4.13}$$

The approximation $p_3(x; 1, \beta)$ is obtained by applying the 48-point Legendre-Gaussian quadrature formula to each term of the convergent alternating series arising from (4.13) when it is divided at successive extrema of $\sin(1+\beta)u$.

5. Further Asymptotic Expansions of $p(x; \alpha, \beta)$

A double series expansion of the product $\exp(-D_1t^\alpha)\sin D_2t^\alpha$ in (4.11), an interchange of integration and summation, and the application of Watson's lemma yields

$$p(x;\alpha,\beta)\sim\frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}}{k!(2l+1)!}D_1^kD_2^{2l+1}\frac{\Gamma[(k+2l+1)\alpha+1]}{x^{(k+2l+1)\alpha+1}}\quad(\alpha\neq 1) \tag{5.1}$$

as $x\to\infty$.

Expanding the integrand of (4.13) yields

$$p(x;1,\beta)=\frac{1}{\pi}\Big[(1+\beta)\int_0^\infty e^{-xt}t\,dt-\frac{2\beta}{\pi}(1+\beta)\int_0^\infty e^{-xt}t^2\ln dt$$

$$-\frac{(1+\beta)}{3!}\int_0^\infty e^{-xt}t^3dt+\left(\frac{2\beta}{\pi}\right)^2(1+\beta)\int_0^\infty e^{-xt}t^3\ln^2 t\,dt+\frac{2\beta}{\pi}(1+\beta)^3$$

$$\times\int_0^\infty e^{-xt}t^4\ln t\,dt-\left(\frac{2\beta}{\pi}\right)^3(1+\beta)\int_0^\infty e^{-xt}t^4\ln^3 t\,dt+\ldots\Big]. \tag{5.2}$$

With the aid of integral tables [74, pp. 576–578] we obtain the incomplete asymptotic expansion

$$p(x;\,1,\,\beta)\sim\frac{1}{\pi}\frac{(1+\beta)}{x^2}-\frac{2^2\beta(1+\beta)}{\pi^2x^3}\,[\psi(3)-\ln x]$$

$$+\left[\frac{12\beta^2}{\pi^2}\,(1+\beta)\{[\psi(4)-\ln x]^2+\zeta(2,3)\}-(1+\beta)^3\right]\frac{1}{x^4}$$

$$+\left[\frac{8(1+\beta)^3\beta}{\pi}\,[\psi(5)-\ln x]-\frac{2^5\beta^3}{\pi^3}\,(1+\beta)\{[\psi(5)-\ln x]^3\right.$$

$$\left.+\,[2\psi(5)-3\ln x]\zeta(2,4)-2\zeta(3,4)\}\right]\frac{1}{x^5},\quad x\to\infty, \tag{5.3}$$

which is valid for $\beta>0$. The symbols ψ and ζ represent the digamma and generalized Riemann zeta functions

$$\psi(z)=\frac{d\ln\Gamma(z)}{dz}\text{ and }\zeta(z,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^z}.$$

6. Preparation of Tables

To tabulate *pdf* values computer programs were prepared from the various approximations. Since values of the *pdf* were both printed and punched, the two sets of values were compared to safeguard against mechanical errors. In the vicinity of the maximum of each *pdf* we employed a variable mesh size in the x direction which depended on $|dp(x;\,\alpha,\,\beta)/dx|$. Hence, for $\alpha\leqslant\frac{1}{2}$, $\Delta x=0.001$, 0.01, or 0.1; for $\alpha=\frac{3}{4}$, $\Delta x=0.01$ or 0.1; and for $\alpha\geqslant 1$, $\Delta x=0.1$. On the *pdf* tails most of the original values with a mesh size $\Delta x=0.1$ were discarded in favor of linear interpolation with steps $\Delta x=1$, 10, or 100. Consequently, after reading all *pdf* values into the computer, we deleted unnecessary values, rounded the remaining ones to four decimal places, and punched the results on cards. After some trial and error the appended printout (reduced in size) was obtained from the punched cards.

7. Checking the Tables

In addition to the previous cross checking, the appropriate approximations, closed forms, and special values were compared. We also made spot checks by using

(a) $p_1(x;\,1,\,\beta)$ over $\beta>0$;

(b) $p_5(x;\,1,\,\beta)$ over $\beta>0$, for large values of x near x_T;

(c) $p_4(x;\,\alpha,\,\beta)$ over $\alpha\neq 1$, $\beta<1$ for large values of x near x_T;

(d) $p_6(x;\,\alpha,\,\beta)$ over $\alpha\neq 1$, for values of x near 0.

Approximation $p_5(x;\,1,\,\beta)$ is the truncation of the asymptotic expansion (5.3), two terms of which were generally required for agreement with the tabulated *pdf* values. The approximations $p_4(x;\,\alpha,\,\beta)$ and $p_6(x;\,\alpha,\,\beta)$ are the truncations of the first and second expansions of section 2.21 respectively. (See sec. 2.22 for asymptotic properties of the two expansions of sec. 2.21.)

By use of the following equations of V. M. Zolotarev [38, p. 166], an additional check is available for $\alpha=\frac{1}{2}$. He showed that

$$p\left(x;\frac{1}{2},\beta\right)=\mathrm{Re}\left\{\frac{z}{\pi x}\left[\pi^{1/2}e^{-z^2}-2iw(z)\right]\right\} \tag{7.1}$$

where

$$w(z)=e^{-z^2}\int_0^z e^{t^2}dt, \qquad z=\frac{1+\beta-i(1-\beta)}{2\sqrt{2}x} \tag{7.2}$$

By setting $w(z)=u(p,-q)+iv(p,-q)$ and $z=p-iq$ we obtain

$$p\left(x;\frac{1}{2},\beta\right)=\frac{1}{\pi^{1/2}}\,(2x)^{-3/2}e^{-\beta/(2x)}\left[(1+\beta)\cos\frac{1-\beta^2}{4x}+(1-\beta)\sin\frac{1-\beta^2}{4x}-\frac{2}{\pi x}\,(qu-pv)\right]. \tag{7.3}$$

Karpov [78] tabulates u and v from $w(\rho e^{i\theta})$. Checking of several tabulated *pdf* values corresponding to $\beta=0.25$ and $\beta=1$ reveals no error in our tables greater than 10^{-4}.

Many of the tabulated values of $p(x;\,1.5,\pm 1)$ were checked with values found in a report by Mandelbrot and Zarnfaller [88]; no discrepancy was found. A check is also available from Fama and Roll's [85] 4-place table of *cdf*'s for $\beta=0$ and $\alpha=1.0(0.1)2.0$; see section 10.

To detect blunders which escaped the preceding checks, *pdf* values from the punched cards were subjected to differencing of orders 1 through 12. Miller [80] states that blunders may occur when sixth and tenth difference magnitudes greater than 22 and 300 respectively appear. Using this technique uncovered several entries which were obviously erroneous.

8. Interpolation

8.1 *Interpolation with respect to* x.

To reduce the size of the table of *pdf* values it is necessary to establish appropriate degree interpolation polynomials over certain intervals of x for each $p(x;\,\alpha,\,\beta)$. We justify this interpolation by noting the analytic properties of $p(x;\,\alpha,\,\beta)$.

For $\alpha\geq 1$, (except $\alpha=1$ and $\beta=0$), $p(x;\,\alpha,\,\beta)$ is an entire function of x[sec. 2.17]. Hence, a sequence of interpolating polynomials $\{p_n(x)\}$ converges uniformly to $p(x;\,\alpha,\,\beta)$ over a given closed interval [71]. For $\alpha<1$, $|dp(x;\,\alpha,\,\beta)/dx|$ becomes large at or near $x=0$. Therefore, interpolation over $[0,\,x_s]$, for some small x_s is difficult. However, away from $x=0$, $p(x;\,\alpha,\,\beta)$ is an analytic function even for $\alpha<1$ [23], so that some $\{p_n(x)\}$ provides interpolation for $p(x;\,\alpha,\,\beta)$ for $x>x_s$.

The Lagrange interpolation polynomial for n equally spaced abscissas x_k and corresponding ordinates f_k is represented by

$$f(x_0+ph)=\sum_k A_k^n(p)f_k+R_{n-1} \tag{7.4}$$

where the index k is defined via the inequalities

$$-\tfrac{1}{2}(n-2)\leq k\leq n/2 \qquad (n \text{ even}),$$

$$-\tfrac{1}{2}(n-1)\leq k\leq \tfrac{1}{2}(n-1) \qquad (n \text{ odd}).$$

The coefficients $A_k^n(p)$ are defined in [69, p. 878] and tabulated in [69, pp. 900–913]. We computed a sample of 150–300 *pdf* values with $\Delta x=0.01$ (in contrast with $\Delta x=0.1$ for the tabulated *pdf* values) around the maximum of each curve for $\alpha\geq 1$ and applied the Lagrange formula

with $n=2(1)8$; $p=0.1, 0.5, 0.9$; $h=0.1$; and sliding intervals beginning with 0.00, 0.01, 0.02, ... The interpolated points were compared with the computed points and all errors (E) were tallied and classified according to the magnitudes:

$$\text{(a) } |E| \geq 10^{-4}, \qquad \text{(b) } 10^{-4} > |E| \geq 0.5(10^{-4}), \qquad \text{(c) } 0.5(10^{-4}) > |E|.$$

An example below (table 2) illustrates this scheme:
$\alpha=1$, $\beta=-0.25$, 154 points starting at $x=0.00$, i.e., $0.00 \leq x \leq 1.53$.

TABLE 2

p	Degree of interpolation	Number of errors in (a)	Interval containing errors in (a)	Number of errors in (b)	Interval containing errors in (b)
0.1	2 to 7[a]	0		0	
0.5	2	8	[0.57, 0.72]	36	[0.45, 0.88]
0.5	3 to 7	0		0	
0.9	2	0		1	
0.9	3 to 7	0		0	

[a] That is, each degree from 2 to 7 was used, and none of the formulas yielded any errors.

The interpolation results above illustrate that for $\alpha=1$ and $\beta=-0.25$ a third-degree polynomial succeeds over [0.00, 1.00]. Furthermore, it appears that second-degree polynomial interpolation is successful for the tabulated *pdf* values from $x=1.0$ to $x=10.0$. An additional examination of the *pdf* values for $\alpha=1$ and $\beta=0.25$ reveals that linear interpolation is applicable beyond $x=10.0$ for $h=1$ and $0<p<1$; i.e., the next table entry corresponds to $x=11$ (instead of $x=10.1$). We summarize our results below in table 3. Notice the range of application means, for instance, in the first line of table 3 that second-degree polynomial interpolation is valid for any three consecutive tabulated *pdf* values from $x=2.0$ to $x=8.0$.

TABLE 3. *Parameters for interpolation with respect to* x *for the tabulated values of* $p(x; \alpha, \beta)$ *in section 13*

Error of interpolation $< 5 \times 10^{-5}$				
α	β	Degree of interpolation	Range of application	
0.25	−1.00(0.25)0.75	2	2.0	8.0
0.50	−1.00(0.25)−0.25	3	2.0	4.0
		2	4.0	10.0
	0.00(0.25)0.75	2	2.0	10.0
0.75	−1.00	5	2.0	4.0
		4	4.0	6.0
		3	6.0	8.0
		2	8.0	13.0
	−0.75	4	2.0	5.0
		3	5.0	7.0
		2	7.0	13.0
	−0.50, −0.25	3	2.0	4.0
		2	4.0	13.0
	0.00(0.25)0.75	2	2.0	13.0
1.00	−1.00	4	0.0	2.3
	−0.75(0.25)0.00	4	0.0	2.0
		3	2.0	4.0
		2	4.0	10.0
	0.25(0.25)1.00	3	0.0	2.0
		2	2.0	10.0
1.25, 1.50, 1.75	All β	2	0.0	8.0

8.2. *Interpolation with respect to* β.

The accuracy of polynomial interpolation with respect to β was estimated from the remainder formulas for Lagrangian interpolation in the Handbook of Mathematical Functions [69, sec. 25.2, pp. 878–879]. The accuracy was also tested by extensive application of Lagrangian interpolation to the tabulated *pdf* values using the interval on β of 0.50 to interpolate the *pdf* values at the midpoints of intervals and comparing with the corresponding tabulated values. We conclude that:

(a) $p(x; \alpha, \beta)$ can be obtained accurate to 3 or 4 decimal places (DP) by at least one of 2-, 3-, or 4-point Lagrangian interpolation with respect to β (using coefficients given in the Handbook [69, table 25.1, pp. 900–903]) for all β for the tabulated α's and any x not too near to 0.0.

(b) Higher than 4-point interpolation may not necessarily improve the accuracy since higher-order differences often do not decrease in magnitude.

(c) $p(0; \alpha, \beta)$ for (α, β) not in the table should always be calculated from the closed formula, section 2.23 (not applicable for $\alpha = 1$).

Rule (a) cannot be easily stated more quantitatively. For sufficiently large x linear interpolation with respect to β yields 4 *DP* accuracy (i.e., within 1 unit in the 4th *DP*). As x decreases, higher-degree interpolation is required for the same accuracy. For example, for $\alpha = 0.25$ linear interpolation yields 4 *DP* accuracy for $x \geq 1.00$ but cubic is in general needed (and is sufficient) for $x = 0.10$. For $\alpha = 1$ quadratic interpolation yields 4 *DP* accuracy for $x \geq 2.0$, but only 3 *DP* can be attained for $x = 0.5$, even if cubic or quartic interpolation is used.

8.3. *Interpolation with respect to* α.

Interpolation with respect to α was investigated in the same way as that with respect to β. Unfortunately the results are mostly negative because there are as many as three local maxima with respect to α for tabulated values of $\beta \neq 0$ and x. We conclude that:

(a) For $\beta = 0$, linear quadratic, or cubic interpolation with respect to α yields 3 *DP* accuracy for $x \geq 1.0$.

(b) For $\beta \neq 0$ quadratic interpolation may yield 2 *DP* or 3 *DP* accuracy for $\alpha > 1.50$, but interpolation is unreliable for $\alpha < 1.50$.

9. Truncation Points (x_T)

In section 4 we defined x_T such that $p(x_T; \alpha, \beta) = 0.00010$ and $p(x_{T-1}; \alpha, \beta) = 0.00011$ with $x_T = x_{T-1} + 0.1$. The last tabulated *pdf* value is rounded to 0.0001, and the tabulated x is usually an integer, not the actual truncation point x_T. We show in table 4 the computed truncation points x_T for every (α, β).

TABLE 4. *Truncation points of* p(x; α, β)

α \ β	−1.00	−0.75	−0.50	−0.25	0.00	0.25	0.50	0.75	1.00	β \ α
0.25	413.2	367.6	321.2	274.0	225.6	175.8	123.6	68.7	.....	.25
0.50	251.2	227.8	203.6	178.3	151.7	123.2	91.9	55.6	.....	.50
0.75	140.9	129.3	117.1	104.2	90.4	75.2	58.0	37.1	.....	.75
1.00	2.3	26.8	39.2	48.4	56.5	63.5	70.1	76.2	81.9	1.00
1.25	49.1	46.5	43.6	40.5	36.9	33.0	28.0	21.2	5.5	1.25
1.50	32.4	30.6	28.8	26.9	24.1	22.3	19.6	14.6	4.8	1.50
1.75	20.3	19.5	18.4	17.4	16.0	14.4	12.6	10.0	5.1	1.75

10. Computation of the Stable (Cumulative) Distribution Function $P(x; \alpha, \beta)$

In a very direct way it is possible to calculate the (cumulative) distribution function $P(x; \alpha, \beta)$ of a stable distribution for the values of α and β in table 1 except for the cases $(\alpha = 1, |\beta| \leq 1)$,

$(\alpha=0.75,\ |\beta|\leq 1)$, $(\alpha>1,\ |\beta|=1)$. Assuming $\alpha\leq\frac{1}{2}$, $|\beta|\leq 1$, we obtain a representation of $P(x;\alpha,\beta)$ from (4.11) by an interchange of the order of integration:

$$P(x;\alpha,\beta)=\int_{-\infty}^{x}p(x';\alpha,\beta)dx',$$

$$=1-\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{1}{t}\,e^{-xt-D_1t^{\alpha}}\right]\sin D_2t^{\alpha}dt\,(x>0). \tag{10.1}$$

Via the identity $p(-x;\alpha,\beta)=p(x;\alpha,\beta)$ we have from (10.1) the representation

$$P(-x;\alpha,\beta)=\frac{1}{\pi}\int_{0}^{\infty}\frac{1}{t}\,e^{-xt-D_1(\alpha,\beta)t^{\alpha}}\sin\left[D_2(\alpha,-\beta)t^{\alpha}\right]dt\,(x>0). \tag{10.2}$$

Hence, an approximation paralleling (4.12) requires only several terms for convergence in the t direction.

To compute $P(x;\alpha,\beta)$ for the category $(\alpha>1,\ |\beta|<1)$ we break up the interval of integrational $(-\infty,x)$ into two subintervals $(-\infty,-10)$ and $(-10,x)$. The contribution to $P(x;\alpha,\beta)$ over the subinterval $(-\infty,-10)$ may be found from the approximation

$$P_4(-10;\alpha,\beta)=\int_{10}^{\infty}p_4(x;\alpha,-\beta)dx=\frac{1}{\pi}\sum_{n=0}^{N}\frac{A_n(\alpha,-\beta)}{\alpha n}\,10^{-\alpha n} \tag{10.3}$$

where $N=4$ usually suffices. Over the second subinterval $(-10,x)$ Simpson's rule is applied to the tabulated *pdf* values by assuming the rounding error $(\pm 5\times 10^{-5})$ is uncorrelated from ordinate to ordinate. To carry this further, let $n/2=100$, $x=10$, and $h=1$ in Simpson's rule,

$$\int_{x_0}^{x_n}p(x;\alpha,\beta)dx=\frac{h}{3}\left[p(x_0;\alpha,\beta)+4p(x_0+h,\alpha,\beta)+2p(x_0+2h;\alpha,\beta)\right.$$

$$\left.+\ \ldots\ +p(x_n,\alpha,\beta)\right]-\frac{nh^5}{180}\frac{d^4}{dx^4}p(x;\alpha,\beta)\,|_{x=\xi}, \tag{10.4}$$

$x_0<\xi<x_n$. In view of the form of (10.4) the total rounding error is less than $\frac{1}{3}\cdot\frac{1}{2}\,(10^{-4})\,(2+100\cdot 4^2+99\cdot 2^2)\,\frac{3}{\sqrt{3}}\doteq 1.3\times 10^{-4}$. The factor $1/\sqrt{3}$ results from the assumption of uniform distribution for individual ordinate rounding errors, and the factor 3 results from defining the "maximum" total rounding error as 3 times the standard deviation.

The integration error term of (10.4) has the bound (sec. 2.18)

$$\left|\frac{nh^5}{180}\frac{d^4}{dx^4}p(\cdot\,x;\alpha,\beta)\right|\leq\frac{nh^5}{180}\frac{1}{\pi\alpha}\,\Gamma\left(\frac{5}{\alpha}\right). \tag{10.5}$$

When $\alpha\geq 1$, the error bound (10.5) is not more than 2×10^{-4} for $h=0.1$ and not more than 2×10^{-9} for $h=0.01$.

Application of Simpson's rule (plus linear interpolation for intermediate x) to the three cases of explicit *pdf*'s (normal, Cauchy, $1/\chi_1^2$) indicates that the above bounds are conservative: (1) The maximum error of 7 normal probabilities calculated was 3×10^{-7}, attributable to rounding. (2) The maximum error of 59 Cauchy probabilities calculated was 8×10^{-5} except in the tail where

the steps in x become 10, and then the maximum error was 3×10^{-4}. (3) The maximum error of twelve $1/\chi_1^2(\alpha=0.5, \beta=-1)$ probabilities was 1.3×10^{-5} for x in steps of 0.01 and 3.6×10^{-5} for x in steps of 0.1.

Values of *cdf* for $\alpha=1.5$, $\beta=0$, as well as the above ones for $\alpha=1$, $\beta=0$ and $\alpha=2$, were derived for comparison with Fama and Roll's [85] 4-place table for *cdf*'s for $\beta=0$ and $\alpha=1.0(0.1)2.0$. In 39 comparisons there was only one difference (3) greater than 1 in the 4th place.

The rather dense grid of values of x, as fine as 0.001 for the largest *pdf* values, is an important contributor to the accuracy of calculation of probabilities. We conclude from the above that probabilities can be calculated by Simpson's rule plus linear interpolation with maximum error of 3×10^{-4} for $\alpha \geq 1$. For $\alpha < 1$ we have no rigorous error bound and the errors tend to be larger than for $\alpha \geq 1$, but calculated probabilities may well have satisfactory accuracies comparable to those above.

The authors thank Garney Hardy [2] and Matthew Lojko [3] of NBS for computer programming support. Stephen Jarvis, Jr., and John Sopka of NBS were especially helpful in their encouragement of this project.

11. Selected Bibliography

A few listed works, Feller's books in particular, may have material in more than one of the following four categories, but each is listed only once. Several Russian papers have English translations in "Selected Translations in Mathematical Statistics and Probability", published for the Institute of Mathematical Statistics by the American Mathematical Society, Providence, Rhode Island. These are identified below as AMS-IMS.

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[89] Nationsl Bureau of Standards, Table of Arctan x, Applied Math. Series 26 (U.S. Government Printing Office, Washington, 1953). *Cdf* for $\alpha = 1$, $\beta = 0$.

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PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.000	7.6394	6.8502	6.8502	4.8079	4.8079	2.2737	2.2737	0.0000	0.000
0.001	5.6948	6.3547	4.4417	6.1773	2.8977	5.0616	1.3461	3.1154	0.001
0.002	4.6299	5.3938	3.5237	5.6279	2.2742	5.1996	1.0566	4.0753	0.002
0.003	3.9662	4.7186	2.9832	5.0931	1.9162	4.9694	0.8910	4.2832	0.003
0.004	3.5002	4.2182	2.6138	4.6482	1.6743	4.6848	0.7793	4.2581	0.004
0.005	3.1493	3.8294	2.3403	4.2802	1.4966	4.4098	0.6972	4.1495	0.005
0.006	2.8730	3.5165	2.1273	3.9721	1.3588	4.1586	0.6336	4.0112	0.006
0.007	2.6483	3.2580	1.9555	3.7103	1.2480	3.9328	0.5824	3.8651	0.007
0.008	2.4609	3.0400	1.8132	3.4848	1.1565	3.7304	0.5401	3.7207	0.008
0.009	2.3018	2.8531	1.6930	3.2882	1.0794	3.5486	0.5044	3.5821	0.009
0.01	2.1646	2.6907	1.5897	3.1152	1.0132	3.3847	0.4738	3.4510	0.01
0.02	1.3915	1.7553	1.0146	2.0814	0.6461	2.3431	0.3036	2.5142	0.02
0.03	1.0443	1.3252	0.7595	1.5870	0.4839	1.8131	0.2281	1.9863	0.03
0.04	0.8419	1.0718	0.6116	1.2907	0.3899	1.4869	0.1842	1.6482	0.04
0.05	0.7078	0.9029	0.5139	1.0912	0.3278	1.2640	0.1551	1.4118	0.05
0.06	0.6118	0.7815	0.4440	0.9469	0.2834	1.1011	0.1343	1.2366	0.06
0.07	0.5394	0.6897	0.3914	0.8373	0.2500	0.9764	0.1186	1.1011	0.07
0.08	0.4828	0.6177	0.3503	0.7509	0.2238	0.8777	0.1063	0.9930	0.08
0.09	0.4371	0.5596	0.3172	0.6811	0.2028	0.7975	0.0964	0.9046	0.09
0.10	0.3995	0.5117	0.2899	0.6233	0.1854	0.7310	0.0882	0.8309	0.10
0.11	0.3680	0.4714	0.2670	0.5747	0.1709	0.6749	0.0814	0.7685	0.11
0.12	0.3411	0.4371	0.2475	0.5332	0.1585	0.6268	0.0755	0.7149	0.12
0.13	0.3179	0.4075	0.2307	0.4974	0.1478	0.5852	0.0705	0.6683	0.13
0.14	0.2977	0.3817	0.2161	0.4661	0.1384	0.5488	0.0661	0.6275	0.14
0.15	0.2800	0.3590	0.2032	0.4385	0.1302	0.5167	0.0622	0.5914	0.15
0.16	0.2642	0.3388	0.1918	0.4140	0.1230	0.4881	0.0587	0.5592	0.16
0.17	0.2501	0.3208	0.1816	0.3921	0.1165	0.4626	0.0557	0.5304	0.17
0.18	0.2375	0.3046	0.1724	0.3724	0.1106	0.4396	0.0529	0.5043	0.18
0.19	0.2261	0.2899	0.1642	0.3546	0.1053	0.4187	0.0504	0.4807	0.19
0.20	0.2157	0.2766	0.1567	0.3384	0.1005	0.3997	0.0481	0.4592	0.20
0.21	0.2062	0.2645	0.1498	0.3236	0.0961	0.3824	0.0460	0.4396	0.21
0.22	0.1975	0.2534	0.1435	0.3101	0.0921	0.3665	0.0441	0.4215	0.22
0.23	0.1895	0.2431	0.1377	0.2976	0.0884	0.3519	0.0424	0.4048	0.23
0.24	0.1822	0.2337	0.1324	0.2861	0.0850	0.3383	0.0408	0.3895	0.24
0.25	0.1754	0.2249	0.1274	0.2754	0.0819	0.3258	0.0392	0.3752	0.25
0.26	0.1690	0.2168	0.1228	0.2655	0.0789	0.3142	0.0379	0.3619	0.26
0.27	0.1631	0.2093	0.1186	0.2562	0.0762	0.3033	0.0366	0.3495	0.27
0.28	0.1576	0.2022	0.1146	0.2476	0.0737	0.2932	0.0353	0.3379	0.28
0.29	0.1525	0.1956	0.1109	0.2396	0.0713	0.2837	0.0342	0.3271	0.29
0.30	0.1477	0.1894	0.1074	0.2320	0.0691	0.2747	0.0331	0.3169	0.30
0.31	0.1431	0.1836	0.1041	0.2249	0.0670	0.2664	0.0321	0.3073	0.31
0.32	0.1389	0.1781	0.1010	0.2182	0.0650	0.2585	0.0312	0.2983	0.32
0.33	0.1348	0.1730	0.0981	0.2119	0.0631	0.2510	0.0303	0.2898	0.33
0.34	0.1310	0.1681	0.0953	0.2059	0.0613	0.2440	0.0295	0.2817	0.34
0.35	0.1274	0.1635	0.0927	0.2003	0.0597	0.2374	0.0287	0.2741	0.35
0.36	0.1240	0.1591	0.0902	0.1949	0.0581	0.2310	0.0279	0.2668	0.36
0.37	0.1208	0.1549	0.0879	0.1899	0.0566	0.2251	0.0272	0.2600	0.37
0.38	0.1177	0.1510	0.0857	0.1850	0.0552	0.2194	0.0265	0.2534	0.38
0.39	0.1148	0.1473	0.0835	0.1804	0.0538	0.2139	0.0259	0.2472	0.39
0.40	0.1120	0.1437	0.0815	0.1761	0.0525	0.2088	0.0253	0.2413	0.40

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.41	0.1094	0.1403	0.0796	0.1719	0.0513	0.2039	0.0247	0.2356	0.41
0.42	0.1069	0.1370	0.0778	0.1679	0.0501	0.1991	0.0241	0.2302	0.42
0.43	0.1044	0.1339	0.0760	0.1641	0.0490	0.1947	0.0236	0.2251	0.43
0.44	0.1021	0.1309	0.0743	0.1605	0.0479	0.1903	0.0231	0.2201	0.44
0.45	0.0999	0.1281	0.0727	0.1570	0.0469	0.1862	0.0226	0.2154	0.45
0.46	0.0978	0.1254	0.0712	0.1536	0.0459	0.1823	0.0221	0.2108	0.46
0.47	0.0957	0.1227	0.0697	0.1504	0.0449	0.1785	0.0216	0.2065	0.47
0.48	0.0938	0.1202	0.0683	0.1474	0.0440	0.1748	0.0212	0.2023	0.48
0.49	0.0919	0.1178	0.0669	0.1444	0.0432	0.1713	0.0208	0.1983	0.49
0.50	0.0901	0.1155	0.0656	0.1416	0.0423	0.1680	0.0204	0.1944	0.50
0.51	0.0883	0.1133	0.0643	0.1388	0.0415	0.1647	0.0200	0.1907	0.51
0.52	0.0867	0.1111	0.0631	0.1362	0.0407	0.1616	0.0196	0.1871	0.52
0.53	0.0850	0.1090	0.0620	0.1336	0.0400	0.1586	0.0193	0.1836	0.53
0.54	0.0835	0.1070	0.0608	0.1312	0.0392	0.1557	0.0189	0.1802	0.54
0.55	0.0820	0.1051	0.0597	0.1288	0.0385	0.1529	0.0186	0.1770	0.55
0.56	0.0805	0.1032	0.0587	0.1265	0.0379	0.1502	0.0183	0.1739	0.56
0.57	0.0791	0.1014	0.0577	0.1243	0.0372	0.1476	0.0179	0.1709	0.57
0.58	0.0778	0.0997	0.0567	0.1222	0.0366	0.1450	0.0176	0.1680	0.58
0.59	0.0765	0.0980	0.0557	0.1201	0.0360	0.1426	0.0174	0.1652	0.59
0.60	0.0752	0.0964	0.0548	0.1181	0.0354	0.1402	0.0171	0.1624	0.60
0.61	0.0740	0.0948	0.0539	0.1162	0.0348	0.1380	0.0168	0.1598	0.61
0.62	0.0728	0.0933	0.0531	0.1143	0.0342	0.1357	0.0165	0.1572	0.62
0.63	0.0716	0.0918	0.0522	0.1125	0.0337	0.1336	0.0163	0.1547	0.63
0.64	0.0705	0.0904	0.0514	0.1107	0.0332	0.1315	0.0160	0.1523	0.64
0.65	0.0694	0.0890	0.0506	0.1090	0.0327	0.1295	0.0158	0.1500	0.65
0.66	0.0684	0.0876	0.0498	0.1074	0.0322	0.1275	0.0155	0.1477	0.66
0.67	0.0673	0.0863	0.0491	0.1058	0.0317	0.1256	0.0153	0.1455	0.67
0.68	0.0663	0.0850	0.0484	0.1042	0.0313	0.1237	0.0151	0.1434	0.68
0.69	0.0654	0.0838	0.0477	0.1027	0.0308	0.1219	0.0149	0.1413	0.69
0.70	0.0644	0.0826	0.0470	0.1012	0.0304	0.1202	0.0147	0.1393	0.70
0.71	0.0635	0.0814	0.0463	0.0998	0.0299	0.1185	0.0145	0.1373	0.71
0.72	0.0626	0.0803	0.0457	0.0984	0.0295	0.1168	0.0143	0.1354	0.72
0.73	0.0618	0.0791	0.0451	0.0970	0.0291	0.1152	0.0141	0.1335	0.73
0.74	0.0609	0.0781	0.0444	0.0957	0.0287	0.1136	0.0139	0.1317	0.74
0.75	0.0601	0.0770	0.0439	0.0944	0.0283	0.1121	0.0137	0.1299	0.75
0.76	0.0593	0.0760	0.0433	0.0931	0.0280	0.1106	0.0135	0.1282	0.76
0.77	0.0585	0.0750	0.0427	0.0919	0.0276	0.1092	0.0133	0.1265	0.77
0.78	0.0578	0.0740	0.0422	0.0907	0.0273	0.1077	0.0132	0.1249	0.78
0.79	0.0570	0.0731	0.0416	0.0896	0.0269	0.1064	0.0130	0.1233	0.79
0.80	0.0563	0.0721	0.0411	0.0884	0.0266	0.1050	0.0128	0.1217	0.80
0.81	0.0556	0.0712	0.0406	0.0873	0.0262	0.1037	0.0127	0.1202	0.81
0.82	0.0549	0.0703	0.0401	0.0862	0.0259	0.1024	0.0125	0.1187	0.82
0.83	0.0543	0.0695	0.0396	0.0852	0.0256	0.1011	0.0124	0.1173	0.83
0.84	0.0536	0.0686	0.0391	0.0841	0.0253	0.0999	0.0122	0.1159	0.84
0.85	0.0530	0.0678	0.0386	0.0831	0.0250	0.0987	0.0121	0.1145	0.85
0.86	0.0523	0.0670	0.0382	0.0821	0.0247	0.0976	0.0119	0.1131	0.86
0.87	0.0517	0.0662	0.0377	0.0812	0.0244	0.0964	0.0118	0.1118	0.87
0.88	0.0511	0.0655	0.0373	0.0802	0.0241	0.0953	0.0117	0.1105	0.88
0.89	0.0505	0.0647	0.0369	0.0793	0.0239	0.0942	0.0115	0.1092	0.89
0.90	0.0500	0.0640	0.0365	0.0784	0.0236	0.0931	0.0114	0.1080	0.90

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.91	0.0494	0.0633	0.0361	0.0775	0.0233	0.0921	0.0113	0.1068	0.91
0.92	0.0489	0.0626	0.0357	0.0767	0.0231	0.0911	0.0112	0.1056	0.92
0.93	0.0483	0.0619	0.0353	0.0758	0.0228	0.0901	0.0110	0.1044	0.93
0.94	0.0478	0.0612	0.0349	0.0750	0.0226	0.0891	0.0109	0.1033	0.94
0.95	0.0473	0.0605	0.0345	0.0742	0.0223	0.0881	0.0108	0.1022	0.95
0.96	0.0468	0.0599	0.0342	0.0734	0.0221	0.0872	0.0107	0.1011	0.96
0.97	0.0463	0.0593	0.0338	0.0726	0.0219	0.0862	0.0106	0.1000	0.97
0.98	0.0458	0.0586	0.0334	0.0718	0.0216	0.0853	0.0105	0.0990	0.98
0.99	0.0453	0.0580	0.0331	0.0711	0.0214	0.0845	0.0104	0.0980	0.99
1.00	0.0449	0.0574	0.0328	0.0704	0.0212	0.0836	0.0103	0.0969	1.00
1.01	0.0444	0.0568	0.0324	0.0696	0.0210	0.0827	0.0102	0.0960	1.01
1.02	0.0440	0.0563	0.0321	0.0689	0.0208	0.0819	0.0101	0.0950	1.02
1.03	0.0435	0.0557	0.0318	0.0683	0.0206	0.0811	0.0100	0.0940	1.03
1.04	0.0431	0.0552	0.0315	0.0676	0.0204	0.0803	0.0099	0.0931	1.04
1.05	0.0427	0.0546	0.0312	0.0669	0.0202	0.0795	0.0098	0.0922	1.05
1.06	0.0423	0.0541	0.0309	0.0663	0.0200	0.0787	0.0097	0.0913	1.06
1.07	0.0418	0.0535	0.0306	0.0656	0.0198	0.0780	0.0096	0.0904	1.07
1.08	0.0414	0.0530	0.0303	0.0650	0.0196	0.0772	0.0095	0.0896	1.08
1.09	0.0410	0.0525	0.0300	0.0644	0.0194	0.0765	0.0094	0.0887	1.09
1.10	0.0407	0.0521	0.0297	0.0638	0.0192	0.0757	0.0093	0.0879	1.10
1.11	0.0403	0.0516	0.0294	0.0632	0.0191	0.0750	0.0092	0.0871	1.11
1.12	0.0399	0.0511	0.0292	0.0626	0.0189	0.0744	0.0092	0.0863	1.12
1.13	0.0396	0.0506	0.0289	0.0620	0.0187	0.0737	0.0091	0.0855	1.13
1.14	0.0392	0.0502	0.0286	0.0615	0.0185	0.0730	0.0090	0.0847	1.14
1.15	0.0388	0.0497	0.0284	0.0609	0.0184	0.0723	0.0089	0.0839	1.15
1.16	0.0385	0.0493	0.0281	0.0604	0.0182	0.0717	0.0088	0.0832	1.16
1.17	0.0382	0.0488	0.0279	0.0598	0.0181	0.0711	0.0087	0.0824	1.17
1.18	0.0378	0.0484	0.0276	0.0593	0.0179	0.0704	0.0087	0.0817	1.18
1.19	0.0375	0.0480	0.0274	0.0588	0.0177	0.0698	0.0086	0.0810	1.19
1.20	0.0372	0.0476	0.0272	0.0583	0.0176	0.0692	0.0085	0.0803	1.20
1.21	0.0369	0.0472	0.0269	0.0578	0.0174	0.0686	0.0085	0.0796	1.21
1.22	0.0365	0.0468	0.0267	0.0573	0.0173	0.0680	0.0084	0.0789	1.22
1.23	0.0362	0.0463	0.0265	0.0568	0.0172	0.0675	0.0083	0.0783	1.23
1.24	0.0359	0.0460	0.0262	0.0563	0.0170	0.0669	0.0082	0.0776	1.24
1.25	0.0356	0.0456	0.0260	0.0558	0.0169	0.0663	0.0082	0.0770	1.25
1.26	0.0353	0.0452	0.0258	0.0554	0.0167	0.0658	0.0081	0.0763	1.26
1.27	0.0350	0.0448	0.0256	0.0549	0.0166	0.0652	0.0080	0.0757	1.27
1.28	0.0348	0.0445	0.0254	0.0545	0.0165	0.0647	0.0080	0.0751	1.28
1.29	0.0345	0.0441	0.0252	0.0540	0.0163	0.0642	0.0079	0.0745	1.29
1.30	0.0342	0.0438	0.0250	0.0536	0.0162	0.0637	0.0078	0.0739	1.30
1.31	0.0339	0.0434	0.0248	0.0532	0.0161	0.0632	0.0078	0.0733	1.31
1.32	0.0337	0.0431	0.0246	0.0527	0.0159	0.0627	0.0077	0.0727	1.32
1.33	0.0334	0.0427	0.0244	0.0523	0.0158	0.0622	0.0077	0.0721	1.33
1.34	0.0331	0.0424	0.0242	0.0519	0.0157	0.0617	0.0076	0.0716	1.34
1.35	0.0329	0.0421	0.0240	0.0515	0.0156	0.0612	0.0076	0.0710	1.35
1.36	0.0326	0.0417	0.0239	0.0511	0.0155	0.0607	0.0075	0.0705	1.36
1.37	0.0324	0.0414	0.0237	0.0507	0.0153	0.0603	0.0074	0.0699	1.37
1.38	0.0321	0.0411	0.0235	0.0504	0.0152	0.0598	0.0074	0.0694	1.38
1.39	0.0319	0.0408	0.0233	0.0500	0.0151	0.0594	0.0073	0.0689	1.39
1.40	0.0317	0.0405	0.0231	0.0496	0.0150	0.0589	0.0073	0.0684	1.40

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
1.41	0.0314	0.0402	0.0230	0.0492	0.0149	0.0585	0.0072	0.0679	1.41
1.42	0.0312	0.0399	0.0228	0.0489	0.0148	0.0580	0.0072	0.0674	1.42
1.43	0.0310	0.0396	0.0226	0.0485	0.0147	0.0576	0.0071	0.0669	1.43
1.44	0.0307	0.0393	0.0225	0.0482	0.0146	0.0572	0.0071	0.0664	1.44
1.45	0.0305	0.0390	0.0223	0.0478	0.0145	0.0568	0.0070	0.0659	1.45
1.46	0.0303	0.0387	0.0222	0.0475	0.0144	0.0564	0.0070	0.0654	1.46
1.47	0.0301	0.0385	0.0220	0.0471	0.0143	0.0560	0.0069	0.0650	1.47
1.48	0.0299	0.0382	0.0219	0.0468	0.0142	0.0556	0.0069	0.0645	1.48
1.49	0.0297	0.0379	0.0217	0.0465	0.0141	0.0552	0.0068	0.0641	1.49
1.50	0.0295	0.0377	0.0215	0.0461	0.0140	0.0548	0.0068	0.0636	1.50
1.51	0.0293	0.0374	0.0214	0.0458	0.0139	0.0544	0.0067	0.0632	1.51
1.52	0.0291	0.0372	0.0212	0.0455	0.0138	0.0540	0.0067	0.0627	1.52
1.53	0.0289	0.0369	0.0211	0.0452	0.0137	0.0537	0.0066	0.0623	1.53
1.54	0.0287	0.0366	0.0210	0.0449	0.0136	0.0533	0.0066	0.0619	1.54
1.55	0.0285	0.0364	0.0208	0.0446	0.0135	0.0529	0.0065	0.0615	1.55
1.56	0.0283	0.0361	0.0207	0.0443	0.0134	0.0526	0.0065	0.0610	1.56
1.57	0.0281	0.0359	0.0205	0.0440	0.0133	0.0522	0.0065	0.0606	1.57
1.58	0.0279	0.0357	0.0204	0.0437	0.0132	0.0519	0.0064	0.0602	1.58
1.59	0.0277	0.0354	0.0203	0.0434	0.0131	0.0515	0.0064	0.0598	1.59
1.60	0.0275	0.0352	0.0201	0.0431	0.0131	0.0512	0.0063	0.0594	1.60
1.61	0.0273	0.0350	0.0200	0.0428	0.0130	0.0509	0.0063	0.0590	1.61
1.62	0.0272	0.0347	0.0199	0.0425	0.0129	0.0505	0.0063	0.0586	1.62
1.63	0.0270	0.0345	0.0197	0.0423	0.0128	0.0502	0.0062	0.0583	1.63
1.64	0.0268	0.0343	0.0196	0.0420	0.0127	0.0499	0.0062	0.0579	1.64
1.65	0.0266	0.0341	0.0195	0.0417	0.0126	0.0496	0.0061	0.0575	1.65
1.66	0.0265	0.0339	0.0194	0.0415	0.0126	0.0492	0.0061	0.0572	1.66
1.67	0.0263	0.0336	0.0192	0.0412	0.0125	0.0489	0.0061	0.0568	1.67
1.68	0.0262	0.0334	0.0191	0.0409	0.0124	0.0486	0.0060	0.0564	1.68
1.69	0.0260	0.0332	0.0190	0.0407	0.0123	0.0483	0.0060	0.0561	1.69
1.70	0.0258	0.0330	0.0189	0.0404	0.0123	0.0480	0.0059	0.0557	1.70
1.71	0.0257	0.0328	0.0188	0.0402	0.0122	0.0477	0.0059	0.0554	1.71
1.72	0.0255	0.0326	0.0187	0.0399	0.0121	0.0474	0.0059	0.0550	1.72
1.73	0.0253	0.0324	0.0185	0.0397	0.0120	0.0471	0.0059	0.0547	1.73
1.74	0.0252	0.0322	0.0184	0.0394	0.0120	0.0468	0.0058	0.0544	1.74
1.75	0.0251	0.0320	0.0183	0.0392	0.0119	0.0466	0.0058	0.0540	1.75
1.76	0.0249	0.0318	0.0182	0.0390	0.0118	0.0463	0.0057	0.0537	1.76
1.77	0.0248	0.0316	0.0181	0.0387	0.0118	0.0460	0.0057	0.0534	1.77
1.78	0.0246	0.0315	0.0180	0.0385	0.0117	0.0457	0.0057	0.0531	1.78
1.79	0.0245	0.0313	0.0179	0.0383	0.0116	0.0455	0.0056	0.0528	1.79
1.80	0.0243	0.0311	0.0178	0.0381	0.0115	0.0452	0.0056	0.0525	1.80
1.81	0.0242	0.0309	0.0177	0.0378	0.0115	0.0449	0.0056	0.0522	1.81
1.82	0.0240	0.0307	0.0176	0.0376	0.0114	0.0447	0.0055	0.0518	1.82
1.83	0.0239	0.0305	0.0175	0.0374	0.0113	0.0444	0.0055	0.0516	1.83
1.84	0.0238	0.0304	0.0174	0.0372	0.0113	0.0441	0.0055	0.0513	1.84
1.85	0.0236	0.0302	0.0173	0.0370	0.0112	0.0439	0.0054	0.0510	1.85
1.86	0.0235	0.0300	0.0172	0.0368	0.0112	0.0436	0.0054	0.0507	1.86
1.87	0.0233	0.0298	0.0171	0.0365	0.0111	0.0434	0.0054	0.0504	1.87
1.88	0.0232	0.0297	0.0170	0.0363	0.0110	0.0432	0.0054	0.0501	1.88
1.89	0.0231	0.0295	0.0169	0.0361	0.0110	0.0429	0.0053	0.0498	1.89
1.90	0.0230	0.0293	0.0168	0.0359	0.0109	0.0427	0.0053	0.0495	1.90

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
1.91	0.0228	0.0292	0.0167	0.0357	0.0108	0.0424	0.0053	0.0493	1.91
1.92	0.0227	0.0290	0.0166	0.0355	0.0108	0.0422	0.0053	0.0490	1.92
1.93	0.0226	0.0289	0.0165	0.0353	0.0107	0.0420	0.0052	0.0487	1.93
1.94	0.0225	0.0287	0.0164	0.0351	0.0107	0.0417	0.0052	0.0485	1.94
1.95	0.0223	0.0285	0.0164	0.0350	0.0106	0.0415	0.0052	0.0482	1.95
1.96	0.0222	0.0284	0.0163	0.0348	0.0106	0.0413	0.0051	0.0479	1.96
1.97	0.0221	0.0282	0.0162	0.0346	0.0105	0.0411	0.0051	0.0477	1.97
1.98	0.0220	0.0281	0.0161	0.0344	0.0104	0.0408	0.0051	0.0474	1.98
1.99	0.0219	0.0279	0.0160	0.0342	0.0104	0.0406	0.0050	0.0472	1.99
2.00	0.0217	0.0278	0.0159	0.0340	0.0103	0.0404	0.0050	0.0469	2.00
2.1	0.0206	0.0264	0.0151	0.0323	0.0098	0.0383	0.0048	0.0445	2.1
2.2	0.0197	0.0251	0.0144	0.0307	0.0094	0.0365	0.0045	0.0423	2.2
2.3	0.0187	0.0239	0.0137	0.0293	0.0089	0.0348	0.0043	0.0404	2.3
2.4	0.0179	0.0229	0.0131	0.0280	0.0085	0.0332	0.0041	0.0386	2.4
2.5	0.0171	0.0219	0.0126	0.0268	0.0082	0.0318	0.0040	0.0369	2.5
2.6	0.0164	0.0210	0.0121	0.0257	0.0078	0.0305	0.0038	0.0354	2.6
2.7	0.0158	0.0201	0.0116	0.0246	0.0075	0.0293	0.0037	0.0340	2.7
2.8	0.0152	0.0194	0.0111	0.0237	0.0072	0.0281	0.0035	0.0327	2.8
2.9	0.0146	0.0186	0.0107	0.0228	0.0070	0.0271	0.0034	0.0314	2.9
3.0	0.0141	0.0180	0.0103	0.0220	0.0067	0.0261	0.0033	0.0303	3.0
3.1	0.0136	0.0174	0.0100	0.0212	0.0065	0.0252	0.0032	0.0292	3.1
3.2	0.0131	0.0168	0.0096	0.0205	0.0063	0.0243	0.0030	0.0282	3.2
3.3	0.0127	0.0162	0.0093	0.0198	0.0061	0.0235	0.0030	0.0273	3.3
3.4	0.0123	0.0157	0.0090	0.0192	0.0059	0.0228	0.0029	0.0264	3.4
3.5	0.0119	0.0152	0.0087	0.0186	0.0057	0.0221	0.0028	0.0256	3.5
3.6	0.0116	0.0148	0.0085	0.0180	0.0055	0.0214	0.0027	0.0248	3.6
3.7	0.0112	0.0143	0.0082	0.0175	0.0054	0.0208	0.0026	0.0241	3.7
3.8	0.0109	0.0139	0.0080	0.0170	0.0052	0.0202	0.0025	0.0234	3.8
3.9	0.0106	0.0135	0.0078	0.0165	0.0051	0.0196	0.0025	0.0228	3.9
4.0	0.0103	0.0131	0.0076	0.0161	0.0049	0.0191	0.0024	0.0221	4.0
4.1	0.0100	0.0128	0.0074	0.0156	0.0048	0.0186	0.0023	0.0215	4.1
4.2	0.0098	0.0125	0.0072	0.0152	0.0047	0.0181	0.0023	0.0210	4.2
4.3	0.0095	0.0121	0.0070	0.0148	0.0046	0.0176	0.0022	0.0204	4.3
4.4	0.0093	0.0118	0.0068	0.0145	0.0045	0.0172	0.0022	0.0199	4.4
4.5	0.0091	0.0115	0.0067	0.0141	0.0043	0.0167	0.0021	0.0194	4.5
4.6	0.0088	0.0113	0.0065	0.0138	0.0042	0.0163	0.0021	0.0190	4.6
4.7	0.0086	0.0110	0.0063	0.0135	0.0041	0.0160	0.0020	0.0185	4.7
4.8	0.0084	0.0108	0.0062	0.0131	0.0040	0.0156	0.0020	0.0181	4.8
4.9	0.0083	0.0105	0.0061	0.0129	0.0040	0.0152	0.0019	0.0177	4.9
5.0	0.0081	0.0103	0.0059	0.0126	0.0039	0.0149	0.0019	0.0173	5.0
5.1	0.0079	0.0101	0.0058	0.0123	0.0038	0.0146	0.0018	0.0169	5.1
5.2	0.0077	0.0098	0.0057	0.0120	0.0037	0.0143	0.0018	0.0166	5.2
5.3	0.0076	0.0096	0.0056	0.0118	0.0036	0.0140	0.0018	0.0162	5.3
5.4	0.0074	0.0095	0.0054	0.0115	0.0036	0.0137	0.0017	0.0159	5.4
5.5	0.0073	0.0093	0.0053	0.0113	0.0035	0.0134	0.0017	0.0156	5.5
5.6	0.0071	0.0091	0.0052	0.0111	0.0034	0.0131	0.0017	0.0152	5.6
5.7	0.0070	0.0089	0.0051	0.0109	0.0034	0.0129	0.0016	0.0149	5.7
5.8	0.0068	0.0087	0.0050	0.0107	0.0033	0.0126	0.0016	0.0147	5.8
5.9	0.0067	0.0086	0.0049	0.0105	0.0032	0.0124	0.0016	0.0144	5.9
6.0	0.0066	0.0084	0.0049	0.0103	0.0032	0.0122	0.0015	0.0141	6.0

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PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
6.1	0.0065	0.0083	0.0048	0.0101	0.0031	0.0120	0.0015	0.0139	6.1
6.2	0.0064	0.0081	0.0047	0.0099	0.0030	0.0117	0.0015	0.0136	6.2
6.3	0.0063	0.0080	0.0046	0.0097	0.0030	0.0115	0.0015	0.0134	6.3
6.4	0.0062	0.0078	0.0045	0.0095	0.0029	0.0113	0.0014	0.0131	6.4
6.5	0.0060	0.0077	0.0044	0.0094	0.0029	0.0111	0.0014	0.0129	6.5
6.6	0.0059	0.0076	0.0044	0.0092	0.0029	0.0109	0.0014	0.0127	6.6
6.7	0.0058	0.0074	0.0043	0.0091	0.0028	0.0108	0.0014	0.0125	6.7
6.8	0.0058	0.0073	0.0042	0.0089	0.0028	0.0106	0.0014	0.0123	6.8
6.9	0.0056	0.0072	0.0042	0.0088	0.0027	0.0104	0.0013	0.0121	6.9
7.0	0.0056	0.0071	0.0041	0.0086	0.0027	0.0103	0.0013	0.0119	7.0
7.1	0.0055	0.0070	0.0040	0.0085	0.0026	0.0101	0.0013	0.0117	7.1
7.2	0.0054	0.0069	0.0040	0.0084	0.0026	0.0099	0.0013	0.0115	7.2
7.3	0.0053	0.0068	0.0039	0.0083	0.0025	0.0098	0.0012	0.0113	7.3
7.4	0.0052	0.0067	0.0038	0.0081	0.0025	0.0096	0.0012	0.0112	7.4
7.5	0.0051	0.0066	0.0038	0.0080	0.0025	0.0095	0.0012	0.0110	7.5
7.6	0.0051	0.0065	0.0037	0.0079	0.0024	0.0093	0.0012	0.0108	7.6
7.7	0.0050	0.0064	0.0037	0.0078	0.0024	0.0092	0.0012	0.0107	7.7
7.8	0.0049	0.0063	0.0036	0.0077	0.0024	0.0091	0.0012	0.0105	7.8
7.9	0.0049	0.0062	0.0036	0.0076	0.0023	0.0089	0.0011	0.0104	7.9
8.0	0.0048	0.0061	0.0035	0.0074	0.0023	0.0088	0.0011	0.0102	8.0
9.	0.0042	0.0053	0.0031	0.0065	0.0020	0.0077	0.0010	0.0090	9.
10.	0.0037	0.0047	0.0027	0.0058	0.0018	0.0069	0.0009	0.0079	10.
11.	0.0034	0.0043	0.0025	0.0052	0.0016	0.0062	0.0008	0.0071	11.
12.	0.0030	0.0039	0.0022	0.0047	0.0015	0.0056	0.0007	0.0065	12.
13.	0.0028	0.0035	0.0020	0.0043	0.0013	0.0051	0.0006	0.0059	13.
14.	0.0025	0.0032	0.0019	0.0040	0.0012	0.0047	0.0006	0.0054	14.
15.	0.0024	0.0030	0.0017	0.0036	0.0011	0.0043	0.0006	0.0050	15.
16.	0.0022	0.0028	0.0016	0.0034	0.0011	0.0040	0.0005	0.0046	16.
17.	0.0020	0.0026	0.0015	0.0032	0.0010	0.0037	0.0005	0.0043	17.
18.	0.0019	0.0024	0.0014	0.0030	0.0009	0.0035	0.0005	0.0041	18.
19.	0.0018	0.0023	0.0013	0.0028	0.0009	0.0033	0.0004	0.0038	19.
20.	0.0017	0.0022	0.0012	0.0026	0.0008	0.0031	0.0004	0.0036	20.
21.	0.0016	0.0020	0.0012	0.0025	0.0008	0.0029	0.0004	0.0034	21.
22.	0.0015	0.0019	0.0011	0.0023	0.0007	0.0028	0.0004	0.0032	22.
23.	0.0014	0.0018	0.0011	0.0022	0.0007	0.0026	0.0003	0.0031	23.
24.	0.0014	0.0017	0.0010	0.0021	0.0007	0.0025	0.0003	0.0029	24.
25.	0.0013	0.0017	0.0010	0.0020	0.0006	0.0024	0.0003	0.0028	25.
26.	0.0013	0.0016	0.0009	0.0019	0.0006	0.0023	0.0003	0.0027	26.
27.	0.0012	0.0015	0.0009	0.0019	0.0006	0.0022	0.0003	0.0025	27.
28.	0.0011	0.0015	0.0008	0.0018	0.0006	0.0021	0.0003	0.0024	28.
29.	0.0011	0.0014	0.0008	0.0017	0.0005	0.0020	0.0003	0.0023	29.
30.	0.0011	0.0014	0.0008	0.0016	0.0005	0.0019	0.0002	0.0023	30.
31.	0.0010	0.0013	0.0008	0.0016	0.0005	0.0019	0.0002	0.0022	31.
32.	0.0010	0.0012	0.0007	0.0015	0.0005	0.0018	0.0002	0.0021	32.
33.	0.0009	0.0012	0.0007	0.0015	0.0005	0.0017	0.0002	0.0020	33.
34.	0.0009	0.0012	0.0007	0.0014	0.0004	0.0017	0.0002	0.0019	34.
35.	0.0009	0.0011	0.0007	0.0014	0.0004	0.0016	0.0002	0.0019	35.
36.	0.0009	0.0011	0.0006	0.0013	0.0004	0.0016	0.0002	0.0018	36.
37.	0.0008	0.0011	0.0006	0.0013	0.0004	0.0015	0.0002	0.0018	37.
38.	0.0008	0.0010	0.0006	0.0012	0.0004	0.0015	0.0002	0.0017	38.

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
48.	0.0006	0.0008	0.0005	0.0009	0.0003	0.0011	0.0001	0.0013	48.
58.	0.0005	0.0006	0.0004	0.0008	0.0002	0.0009		0.0010	58.
68.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0007		0.0009	68.
78.	0.0004	0.0004	0.0003	0.0005	0.0002	0.0006		0.0007	78.
88.	0.0003	0.0004	0.0002	0.0005	0.0001	0.0005		0.0006	88.
98.	0.0003	0.0003	0.0002	0.0004		0.0005		0.0006	98.
108.	0.0002	0.0003	0.0002	0.0004		0.0004		0.0005	108.
118.	0.0002	0.0003	0.0002	0.0003		0.0004		0.0004	118.
128.	0.0002	0.0002	0.0001	0.0003		0.0003		0.0004	128.
138.	0.0002	0.0002		0.0003		0.0003		0.0004	138.
238.	0.0001	0.0001		0.0001		0.0002		0.0002	238.
338.						0.0001		0.0001	338.

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.000	0.6366	0.5287	0.5287	0.3056	0.3056	0.1141	0.1141	0.0000	0.000
0.001	0.6366	0.5315	0.5259	0.3079	0.3032	0.1149	0.1132	0.0000	0.001
0.002	0.6365	0.5343	0.5231	0.3103	0.3009	0.1158	0.1124	0.0000	0.002
0.003	0.6363	0.5371	0.5202	0.3128	0.2987	0.1167	0.1116	0.0000	0.003
0.004	0.6360	0.5398	0.5174	0.3153	0.2965	0.1176	0.1108	0.0000	0.004
0.005	0.6357	0.5426	0.5146	0.3178	0.2943	0.1185	0.1100	0.0000	0.005
0.006	0.6353	0.5453	0.5118	0.3203	0.2921	0.1194	0.1093	0.0000	0.006
0.007	0.6348	0.5480	0.5090	0.3229	0.2900	0.1204	0.1085	0.0000	0.007
0.008	0.6342	0.5507	0.5062	0.3256	0.2879	0.1213	0.1078	0.0000	0.008
0.009	0.6336	0.5534	0.5034	0.3283	0.2859	0.1223	0.1071	0.0000	0.009
0.01	0.6329	0.5560	0.5006	0.3310	0.2838	0.1233	0.1064	0.0000	0.01
0.02	0.6226	0.5799	0.4735	0.3605	0.2651	0.1348	0.0997	0.0000	0.02
0.03	0.6077	0.5981	0.4481	0.3935	0.2487	0.1497	0.0940	0.0000	0.03
0.04	0.5900	0.6096	0.4246	0.4278	0.2343	0.1697	0.0890	0.0002	0.04
0.05	0.5709	0.6150	0.4030	0.4605	0.2215	0.1956	0.0845	0.0016	0.05
0.06	0.5513	0.6152	0.3832	0.4897	0.2101	0.2268	0.0805	0.0065	0.06
0.07	0.5318	0.6112	0.3651	0.5141	0.1998	0.2611	0.0768	0.0170	0.07
0.08	0.5127	0.6042	0.3484	0.5335	0.1905	0.2964	0.0735	0.0340	0.08
0.09	0.4942	0.5950	0.3331	0.5482	0.1820	0.3307	0.0705	0.0571	0.09
0.10	0.4764	0.5841	0.3190	0.5585	0.1743	0.3628	0.0678	0.0850	0.10
0.11	0.4595	0.5722	0.3059	0.5651	0.1672	0.3918	0.0652	0.1161	0.11
0.12	0.4434	0.5597	0.2938	0.5685	0.1606	0.4174	0.0629	0.1488	0.12
0.13	0.4281	0.5467	0.2825	0.5692	0.1545	0.4395	0.0607	0.1818	0.13
0.14	0.4136	0.5336	0.2720	0.5677	0.1489	0.4582	0.0586	0.2141	0.14
0.15	0.3999	0.5205	0.2622	0.5645	0.1437	0.4736	0.0567	0.2450	0.15
0.16	0.3869	0.5075	0.2531	0.5599	0.1388	0.4862	0.0549	0.2739	0.16
0.17	0.3745	0.4948	0.2445	0.5541	0.1342	0.4960	0.0533	0.3005	0.17
0.18	0.3628	0.4823	0.2365	0.5475	0.1299	0.5035	0.0517	0.3248	0.18
0.19	0.3517	0.4701	0.2289	0.5402	0.1259	0.5090	0.0502	0.3466	0.19
0.20	0.3411	0.4582	0.2218	0.5323	0.1221	0.5126	0.0488	0.3661	0.20
0.21	0.3311	0.4467	0.2150	0.5241	0.1185	0.5147	0.0475	0.3833	0.21
0.22	0.3216	0.4355	0.2086	0.5156	0.1151	0.5154	0.0462	0.3983	0.22
0.23	0.3125	0.4248	0.2026	0.5069	0.1119	0.5149	0.0450	0.4113	0.23
0.24	0.3039	0.4143	0.1969	0.4981	0.1088	0.5134	0.0439	0.4225	0.24
0.25	0.2956	0.4043	0.1915	0.4893	0.1059	0.5111	0.0428	0.4319	0.25

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.26	0.2878	0.3946	0.1863	0.4805	0.1032	0.5080	0.0418	0.4398	0.26
0.27	0.2803	0.3852	0.1814	0.4718	0.1006	0.5044	0.0408	0.4463	0.27
0.28	0.2731	0.3762	0.1767	0.4631	0.0981	0.5002	0.0399	0.4515	0.28
0.29	0.2663	0.3674	0.1723	0.4546	0.0957	0.4956	0.0390	0.4555	0.29
0.30	0.2597	0.3590	0.1680	0.4461	0.0935	0.4907	0.0381	0.4586	0.30
0.31	0.2534	0.3509	0.1640	0.4378	0.0913	0.4854	0.0373	0.4607	0.31
0.32	0.2474	0.3431	0.1601	0.4297	0.0892	0.4800	0.0365	0.4619	0.32
0.33	0.2417	0.3356	0.1564	0.4217	0.0872	0.4743	0.0357	0.4625	0.33
0.34	0.2361	0.3283	0.1528	0.4139	0.0853	0.4685	0.0350	0.4624	0.34
0.35	0.2308	0.3212	0.1494	0.4062	0.0835	0.4627	0.0343	0.4617	0.35
0.36	0.2257	0.3144	0.1461	0.3988	0.0817	0.4567	0.0336	0.4605	0.36
0.37	0.2208	0.3079	0.1430	0.3915	0.0801	0.4507	0.0330	0.4589	0.37
0.38	0.2161	0.3015	0.1399	0.3843	0.0784	0.4447	0.0324	0.4569	0.38
0.39	0.2115	0.2954	0.1370	0.3774	0.0769	0.4386	0.0317	0.4545	0.39
0.40	0.2071	0.2895	0.1342	0.3706	0.0753	0.4326	0.0312	0.4518	0.40
0.41	0.2029	0.2838	0.1315	0.3640	0.0739	0.4266	0.0306	0.4489	0.41
0.42	0.1988	0.2782	0.1289	0.3576	0.0725	0.4207	0.0301	0.4457	0.42
0.43	0.1949	0.2728	0.1264	0.3513	0.0711	0.4148	0.0295	0.4423	0.43
0.44	0.1911	0.2676	0.1239	0.3451	0.0698	0.4089	0.0290	0.4387	0.44
0.45	0.1874	0.2626	0.1216	0.3392	0.0686	0.4031	0.0285	0.4351	0.45
0.46	0.1838	0.2577	0.1193	0.3334	0.0673	0.3974	0.0280	0.4312	0.46
0.47	0.1804	0.2530	0.1171	0.3277	0.0662	0.3918	0.0276	0.4273	0.47
0.48	0.1771	0.2484	0.1150	0.3222	0.0650	0.3862	0.0271	0.4233	0.48
0.49	0.1739	0.2440	0.1130	0.3168	0.0639	0.3807	0.0267	0.4192	0.49
0.50	0.1708	0.2397	0.1110	0.3115	0.0628	0.3753	0.0263	0.4151	0.50
0.51	0.1677	0.2355	0.1091	0.3064	0.0618	0.3700	0.0259	0.4109	0.51
0.52	0.1648	0.2314	0.1072	0.3014	0.0608	0.3648	0.0255	0.4067	0.52
0.53	0.1620	0.2275	0.1054	0.2966	0.0598	0.3596	0.0251	0.4025	0.53
0.54	0.1592	0.2236	0.1036	0.2918	0.0589	0.3546	0.0247	0.3983	0.54
0.55	0.1566	0.2199	0.1020	0.2872	0.0579	0.3496	0.0244	0.3941	0.55
0.56	0.1540	0.2163	0.1003	0.2827	0.0570	0.3447	0.0240	0.3898	0.56
0.57	0.1515	0.2128	0.0987	0.2783	0.0562	0.3400	0.0237	0.3856	0.57
0.58	0.1490	0.2094	0.0971	0.2740	0.0553	0.3353	0.0233	0.3814	0.58
0.59	0.1466	0.2060	0.0956	0.2698	0.0545	0.3307	0.0230	0.3772	0.59
0.60	0.1443	0.2028	0.0942	0.2658	0.0537	0.3261	0.0227	0.3731	0.60
0.61	0.1421	0.1996	0.0927	0.2618	0.0529	0.3217	0.0224	0.3689	0.61
0.62	0.1399	0.1966	0.0913	0.2579	0.0522	0.3173	0.0221	0.3648	0.62
0.63	0.1378	0.1936	0.0900	0.2541	0.0514	0.3131	0.0218	0.3608	0.63
0.64	0.1357	0.1907	0.0887	0.2504	0.0507	0.3089	0.0215	0.3567	0.64
0.65	0.1337	0.1878	0.0874	0.2468	0.0500	0.3048	0.0212	0.3527	0.65
0.66	0.1317	0.1851	0.0861	0.2433	0.0493	0.3007	0.0209	0.3488	0.66
0.67	0.1298	0.1823	0.0849	0.2398	0.0486	0.2968	0.0206	0.3449	0.67
0.68	0.1279	0.1797	0.0837	0.2364	0.0480	0.2929	0.0204	0.3410	0.68
0.69	0.1261	0.1771	0.0825	0.2331	0.0473	0.2891	0.0201	0.3372	0.69
0.70	0.1243	0.1746	0.0814	0.2299	0.0467	0.2854	0.0199	0.3335	0.70
0.71	0.1226	0.1722	0.0803	0.2268	0.0461	0.2817	0.0196	0.3297	0.71
0.72	0.1209	0.1698	0.0792	0.2237	0.0455	0.2781	0.0194	0.3261	0.72
0.73	0.1192	0.1675	0.0782	0.2207	0.0449	0.2746	0.0192	0.3224	0.73
0.74	0.1176	0.1652	0.0772	0.2177	0.0444	0.2712	0.0189	0.3189	0.74
0.75	0.1161	0.1630	0.0762	0.2149	0.0438	0.2678	0.0187	0.3153	0.75

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.76	0.1145	0.1608	0.0752	0.2120	0.0433	0.2645	0.0185	0.3119	0.76
0.77	0.1130	0.1587	0.0742	0.2093	0.0427	0.2612	0.0183	0.3084	0.77
0.78	0.1116	0.1566	0.0733	0.2066	0.0422	0.2580	0.0181	0.3051	0.78
0.79	0.1101	0.1546	0.0724	0.2039	0.0417	0.2549	0.0179	0.3017	0.79
0.80	0.1087	0.1526	0.0715	0.2014	0.0412	0.2518	0.0177	0.2984	0.80
0.81	0.1074	0.1506	0.0706	0.1988	0.0408	0.2488	0.0175	0.2952	0.81
0.82	0.1060	0.1487	0.0697	0.1964	0.0403	0.2458	0.0173	0.2920	0.82
0.83	0.1047	0.1469	0.0689	0.1939	0.0398	0.2429	0.0171	0.2888	0.83
0.84	0.1034	0.1450	0.0681	0.1915	0.0394	0.2400	0.0169	0.2857	0.84
0.85	0.1022	0.1433	0.0673	0.1892	0.0389	0.2372	0.0167	0.2827	0.85
0.86	0.1009	0.1415	0.0665	0.1869	0.0385	0.2345	0.0165	0.2797	0.86
0.87	0.0997	0.1398	0.0657	0.1847	0.0380	0.2318	0.0164	0.2767	0.87
0.88	0.0986	0.1381	0.0650	0.1825	0.0376	0.2291	0.0162	0.2738	0.88
0.89	0.0974	0.1365	0.0642	0.1803	0.0372	0.2265	0.0160	0.2709	0.89
0.90	0.0963	0.1349	0.0635	0.1782	0.0368	0.2240	0.0159	0.2681	0.90
0.91	0.0951	0.1333	0.0628	0.1762	0.0364	0.2215	0.0157	0.2653	0.91
0.92	0.0941	0.1318	0.0621	0.1741	0.0360	0.2190	0.0156	0.2625	0.92
0.93	0.0930	0.1303	0.0614	0.1722	0.0356	0.2166	0.0154	0.2598	0.93
0.94	0.0920	0.1288	0.0607	0.1702	0.0353	0.2142	0.0152	0.2572	0.94
0.95	0.0909	0.1273	0.0601	0.1683	0.0349	0.2118	0.0151	0.2545	0.95
0.96	0.0899	0.1259	0.0594	0.1664	0.0346	0.2095	0.0149	0.2519	0.96
0.97	0.0889	0.1245	0.0588	0.1646	0.0342	0.2073	0.0148	0.2494	0.97
0.98	0.0880	0.1231	0.0582	0.1628	0.0338	0.2051	0.0147	0.2469	0.98
0.99	0.0870	0.1218	0.0576	0.1610	0.0335	0.2029	0.0145	0.2444	0.99
1.00	0.0861	0.1205	0.0570	0.1592	0.0332	0.2007	0.0144	0.2420	1.00
1.01	0.0852	0.1192	0.0564	0.1575	0.0328	0.1986	0.0142	0.2396	1.01
1.02	0.0843	0.1179	0.0558	0.1558	0.0325	0.1965	0.0141	0.2372	1.02
1.03	0.0834	0.1166	0.0553	0.1542	0.0322	0.1945	0.0140	0.2349	1.03
1.04	0.0825	0.1154	0.0547	0.1526	0.0319	0.1925	0.0139	0.2326	1.04
1.05	0.0817	0.1142	0.0541	0.1510	0.0316	0.1905	0.0137	0.2303	1.05
1.06	0.0809	0.1130	0.0536	0.1494	0.0313	0.1886	0.0136	0.2281	1.06
1.07	0.0801	0.1118	0.0531	0.1479	0.0310	0.1867	0.0135	0.2259	1.07
1.08	0.0792	0.1107	0.0526	0.1463	0.0307	0.1848	0.0133	0.2237	1.08
1.09	0.0785	0.1096	0.0521	0.1449	0.0304	0.1830	0.0132	0.2216	1.09
1.10	0.0777	0.1085	0.0516	0.1434	0.0301	0.1812	0.0131	0.2195	1.10
1.11	0.0769	0.1074	0.0511	0.1420	0.0299	0.1794	0.0130	0.2174	1.11
1.12	0.0762	0.1063	0.0506	0.1405	0.0296	0.1776	0.0129	0.2154	1.12
1.13	0.0754	0.1053	0.0501	0.1391	0.0293	0.1759	0.0128	0.2134	1.13
1.14	0.0747	0.1042	0.0496	0.1378	0.0291	0.1742	0.0127	0.2114	1.14
1.15	0.0740	0.1032	0.0492	0.1364	0.0288	0.1725	0.0126	0.2094	1.15
1.16	0.0733	0.1022	0.0487	0.1351	0.0285	0.1708	0.0125	0.2075	1.16
1.17	0.0726	0.1012	0.0483	0.1338	0.0283	0.1692	0.0123	0.2056	1.17
1.18	0.0719	0.1003	0.0478	0.1325	0.0280	0.1676	0.0122	0.2037	1.18
1.19	0.0713	0.0993	0.0474	0.1313	0.0278	0.1660	0.0122	0.2019	1.19
1.20	0.0706	0.0984	0.0470	0.1300	0.0276	0.1645	0.0121	0.2001	1.20
1.21	0.0699	0.0975	0.0466	0.1288	0.0273	0.1630	0.0120	0.1983	1.21
1.22	0.0693	0.0966	0.0462	0.1276	0.0271	0.1615	0.0119	0.1965	1.22
1.23	0.0687	0.0957	0.0458	0.1264	0.0269	0.1600	0.0118	0.1948	1.23
1.24	0.0681	0.0948	0.0454	0.1253	0.0266	0.1585	0.0117	0.1930	1.24
1.25	0.0675	0.0940	0.0450	0.1241	0.0264	0.1571	0.0116	0.1913	1.25

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
1.26	0.0669	0.0931	0.0446	0.1230	0.0262	0.1557	0.0115	0.1897	1.26
1.27	0.0663	0.0923	0.0442	0.1219	0.0260	0.1543	0.0114	0.1880	1.27
1.28	0.0657	0.0914	0.0438	0.1208	0.0258	0.1529	0.0113	0.1864	1.28
1.29	0.0651	0.0906	0.0435	0.1197	0.0256	0.1515	0.0112	0.1848	1.29
1.30	0.0646	0.0898	0.0431	0.1186	0.0254	0.1502	0.0111	0.1832	1.30
1.31	0.0640	0.0891	0.0427	0.1176	0.0252	0.1489	0.0110	0.1816	1.31
1.32	0.0635	0.0883	0.0424	0.1166	0.0250	0.1476	0.0110	0.1801	1.32
1.33	0.0629	0.0875	0.0420	0.1156	0.0248	0.1463	0.0109	0.1786	1.33
1.34	0.0624	0.0868	0.0417	0.1146	0.0246	0.1451	0.0108	0.1771	1.34
1.35	0.0619	0.0860	0.0414	0.1136	0.0244	0.1438	0.0107	0.1756	1.35
1.36	0.0614	0.0853	0.0410	0.1126	0.0242	0.1426	0.0106	0.1741	1.36
1.37	0.0608	0.0846	0.0407	0.1116	0.0240	0.1414	0.0106	0.1727	1.37
1.38	0.0603	0.0839	0.0404	0.1107	0.0238	0.1402	0.0105	0.1713	1.38
1.39	0.0599	0.0832	0.0400	0.1097	0.0236	0.1390	0.0104	0.1699	1.39
1.40	0.0594	0.0825	0.0397	0.1088	0.0235	0.1379	0.0103	0.1685	1.40
1.41	0.0589	0.0818	0.0394	0.1079	0.0233	0.1367	0.0103	0.1671	1.41
1.42	0.0584	0.0811	0.0391	0.1070	0.0231	0.1356	0.0102	0.1658	1.42
1.43	0.0580	0.0805	0.0388	0.1062	0.0229	0.1345	0.0101	0.1645	1.43
1.44	0.0575	0.0798	0.0385	0.1053	0.0228	0.1334	0.0100	0.1631	1.44
1.45	0.0570	0.0792	0.0382	0.1044	0.0226	0.1323	0.0100	0.1618	1.45
1.46	0.0566	0.0785	0.0379	0.1036	0.0224	0.1312	0.0099	0.1606	1.46
1.47	0.0562	0.0779	0.0376	0.1028	0.0223	0.1302	0.0098	0.1593	1.47
1.48	0.0557	0.0773	0.0373	0.1019	0.0221	0.1292	0.0098	0.1580	1.48
1.49	0.0553	0.0767	0.0371	0.1011	0.0219	0.1281	0.0097	0.1568	1.49
1.50	0.0549	0.0761	0.0368	0.1003	0.0218	0.1271	0.0096	0.1556	1.50
1.51	0.0544	0.0755	0.0365	0.0995	0.0216	0.1261	0.0096	0.1544	1.51
1.52	0.0540	0.0749	0.0363	0.0988	0.0215	0.1251	0.0095	0.1532	1.52
1.53	0.0536	0.0743	0.0360	0.0980	0.0213	0.1242	0.0094	0.1520	1.53
1.54	0.0532	0.0738	0.0357	0.0972	0.0212	0.1232	0.0094	0.1509	1.54
1.55	0.0528	0.0732	0.0355	0.0965	0.0210	0.1223	0.0093	0.1497	1.55
1.56	0.0524	0.0727	0.0352	0.0957	0.0209	0.1213	0.0092	0.1486	1.56
1.57	0.0521	0.0721	0.0350	0.0950	0.0207	0.1204	0.0092	0.1475	1.57
1.58	0.0517	0.0716	0.0347	0.0943	0.0206	0.1195	0.0091	0.1464	1.58
1.59	0.0513	0.0710	0.0345	0.0936	0.0205	0.1186	0.0091	0.1453	1.59
1.60	0.0509	0.0705	0.0342	0.0929	0.0203	0.1177	0.0090	0.1442	1.60
1.61	0.0506	0.0700	0.0340	0.0922	0.0202	0.1168	0.0089	0.1431	1.61
1.62	0.0502	0.0695	0.0337	0.0915	0.0200	0.1159	0.0089	0.1421	1.62
1.63	0.0498	0.0690	0.0335	0.0908	0.0199	0.1151	0.0088	0.1411	1.63
1.64	0.0495	0.0685	0.0333	0.0902	0.0198	0.1142	0.0088	0.1400	1.64
1.65	0.0491	0.0680	0.0331	0.0895	0.0197	0.1134	0.0087	0.1390	1.65
1.66	0.0488	0.0675	0.0328	0.0889	0.0195	0.1126	0.0087	0.1380	1.66
1.67	0.0485	0.0670	0.0326	0.0882	0.0194	0.1118	0.0086	0.1370	1.67
1.68	0.0481	0.0665	0.0324	0.0876	0.0193	0.1109	0.0086	0.1360	1.68
1.69	0.0478	0.0661	0.0322	0.0869	0.0191	0.1101	0.0085	0.1351	1.69
1.70	0.0475	0.0656	0.0319	0.0863	0.0190	0.1094	0.0084	0.1341	1.70
1.71	0.0471	0.0651	0.0317	0.0857	0.0189	0.1086	0.0084	0.1332	1.71
1.72	0.0468	0.0647	0.0315	0.0851	0.0188	0.1078	0.0083	0.1322	1.72
1.73	0.0465	0.0642	0.0313	0.0845	0.0186	0.1071	0.0083	0.1313	1.73
1.74	0.0462	0.0638	0.0311	0.0839	0.0185	0.1063	0.0082	0.1304	1.74
1.75	0.0459	0.0634	0.0309	0.0833	0.0184	0.1056	0.0082	0.1295	1.75

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
1.76	0.0455	0.0629	0.0307	0.0827	0.0183	0.1048	0.0081	0.1286	1.76
1.77	0.0453	0.0625	0.0305	0.0822	0.0182	0.1041	0.0081	0.1277	1.77
1.78	0.0450	0.0621	0.0303	0.0816	0.0181	0.1034	0.0080	0.1268	1.78
1.79	0.0447	0.0616	0.0301	0.0811	0.0180	0.1027	0.0080	0.1260	1.79
1.80	0.0444	0.0612	0.0299	0.0805	0.0179	0.1020	0.0079	0.1251	1.80
1.81	0.0441	0.0608	0.0297	0.0800	0.0177	0.1013	0.0079	0.1243	1.81
1.82	0.0438	0.0604	0.0296	0.0794	0.0176	0.1006	0.0079	0.1234	1.82
1.83	0.0435	0.0600	0.0294	0.0789	0.0175	0.0999	0.0078	0.1226	1.83
1.84	0.0432	0.0596	0.0292	0.0784	0.0174	0.0992	0.0078	0.1218	1.84
1.85	0.0430	0.0592	0.0290	0.0778	0.0173	0.0986	0.0077	0.1210	1.85
1.86	0.0427	0.0589	0.0288	0.0773	0.0172	0.0979	0.0077	0.1202	1.86
1.87	0.0424	0.0585	0.0287	0.0768	0.0171	0.0973	0.0076	0.1194	1.87
1.88	0.0421	0.0581	0.0285	0.0763	0.0170	0.0966	0.0076	0.1186	1.88
1.89	0.0419	0.0577	0.0283	0.0758	0.0169	0.0960	0.0076	0.1178	1.89
1.90	0.0416	0.0574	0.0281	0.0753	0.0168	0.0954	0.0075	0.1171	1.90
1.91	0.0414	0.0570	0.0280	0.0748	0.0167	0.0948	0.0075	0.1163	1.91
1.92	0.0411	0.0566	0.0278	0.0744	0.0166	0.0941	0.0074	0.1156	1.92
1.93	0.0408	0.0563	0.0276	0.0739	0.0165	0.0935	0.0074	0.1148	1.93
1.94	0.0406	0.0559	0.0275	0.0734	0.0164	0.0929	0.0073	0.1141	1.94
1.95	0.0403	0.0556	0.0273	0.0729	0.0163	0.0923	0.0073	0.1134	1.95
1.96	0.0401	0.0552	0.0271	0.0725	0.0162	0.0917	0.0073	0.1126	1.96
1.97	0.0399	0.0549	0.0270	0.0720	0.0161	0.0912	0.0072	0.1119	1.97
1.98	0.0396	0.0545	0.0268	0.0716	0.0161	0.0906	0.0072	0.1112	1.98
1.99	0.0394	0.0542	0.0267	0.0711	0.0160	0.0900	0.0071	0.1105	1.99
2.00	0.0391	0.0539	0.0265	0.0707	0.0159	0.0895	0.0071	0.1098	2.00
2.1	0.0369	0.0507	0.0250	0.0665	0.0150	0.0841	0.0067	0.1033	2.1
2.2	0.0349	0.0479	0.0237	0.0627	0.0142	0.0793	0.0064	0.0974	2.2
2.3	0.0331	0.0453	0.0225	0.0593	0.0135	0.0749	0.0061	0.0920	2.3
2.4	0.0314	0.0430	0.0214	0.0562	0.0129	0.0709	0.0058	0.0871	2.4
2.5	0.0298	0.0408	0.0204	0.0533	0.0123	0.0673	0.0055	0.0826	2.5
2.6	0.0284	0.0389	0.0194	0.0507	0.0117	0.0640	0.0053	0.0785	2.6
2.7	0.0271	0.0370	0.0186	0.0483	0.0112	0.0609	0.0051	0.0747	2.7
2.8	0.0259	0.0354	0.0178	0.0461	0.0108	0.0581	0.0049	0.0712	2.8
2.9	0.0248	0.0338	0.0170	0.0440	0.0103	0.0554	0.0047	0.0680	2.9
3.0	0.0238	0.0324	0.0163	0.0421	0.0099	0.0530	0.0045	0.0650	3.0
3.1	0.0228	0.0311	0.0157	0.0404	0.0095	0.0508	0.0044	0.0622	3.1
3.2	0.0219	0.0298	0.0151	0.0387	0.0092	0.0487	0.0042	0.0596	3.2
3.3	0.0211	0.0286	0.0145	0.0372	0.0089	0.0467	0.0040	0.0572	3.3
3.4	0.0203	0.0275	0.0140	0.0357	0.0086	0.0449	0.0039	0.0549	3.4
3.5	0.0196	0.0265	0.0135	0.0344	0.0083	0.0432	0.0038	0.0528	3.5
3.6	0.0189	0.0256	0.0130	0.0331	0.0080	0.0415	0.0036	0.0508	3.6
3.7	0.0182	0.0247	0.0126	0.0319	0.0077	0.0400	0.0035	0.0490	3.7
3.8	0.0176	0.0238	0.0122	0.0308	0.0075	0.0386	0.0034	0.0472	3.8
3.9	0.0170	0.0230	0.0118	0.0298	0.0072	0.0373	0.0033	0.0456	3.9
4.0	0.0165	0.0223	0.0114	0.0288	0.0070	0.0360	0.0032	0.0440	4.0
4.1	0.0160	0.0216	0.0111	0.0278	0.0068	0.0348	0.0031	0.0425	4.1
4.2	0.0155	0.0209	0.0107	0.0270	0.0066	0.0337	0.0030	0.0412	4.2
4.3	0.0150	0.0203	0.0104	0.0261	0.0064	0.0326	0.0030	0.0398	4.3
4.4	0.0146	0.0196	0.0101	0.0253	0.0062	0.0316	0.0029	0.0386	4.4
4.5	0.0142	0.0191	0.0098	0.0246	0.0061	0.0307	0.0028	0.0374	4.5

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
4.6	0.0138	0.0185	0.0096	0.0238	0.0059	0.0298	0.0027	0.0363	4.6
4.7	0.0134	0.0180	0.0093	0.0232	0.0058	0.0289	0.0027	0.0352	4.7
4.8	0.0130	0.0175	0.0091	0.0225	0.0056	0.0281	0.0026	0.0342	4.8
4.9	0.0127	0.0170	0.0088	0.0219	0.0055	0.0273	0.0025	0.0332	4.9
5.0	0.0123	0.0166	0.0086	0.0213	0.0053	0.0265	0.0025	0.0323	5.0
5.1	0.0120	0.0161	0.0084	0.0207	0.0052	0.0258	0.0024	0.0314	5.1
5.2	0.0117	0.0157	0.0082	0.0202	0.0051	0.0251	0.0023	0.0306	5.2
5.3	0.0114	0.0153	0.0080	0.0197	0.0049	0.0245	0.0023	0.0298	5.3
5.4	0.0112	0.0149	0.0078	0.0192	0.0048	0.0238	0.0022	0.0290	5.4
5.5	0.0109	0.0146	0.0076	0.0187	0.0047	0.0232	0.0022	0.0282	5.5
5.6	0.0106	0.0142	0.0074	0.0182	0.0046	0.0227	0.0021	0.0275	5.6
5.7	0.0104	0.0139	0.0073	0.0178	0.0045	0.0221	0.0021	0.0269	5.7
5.8	0.0102	0.0136	0.0071	0.0174	0.0044	0.0216	0.0020	0.0262	5.8
5.9	0.0099	0.0133	0.0069	0.0170	0.0043	0.0211	0.0020	0.0256	5.9
6.0	0.0097	0.0130	0.0068	0.0166	0.0042	0.0206	0.0020	0.0250	6.0
6.1	0.0095	0.0127	0.0067	0.0162	0.0041	0.0201	0.0019	0.0244	6.1
6.2	0.0093	0.0124	0.0065	0.0158	0.0041	0.0197	0.0019	0.0238	6.2
6.3	0.0091	0.0121	0.0064	0.0155	0.0040	0.0192	0.0018	0.0233	6.3
6.4	0.0089	0.0119	0.0063	0.0152	0.0039	0.0188	0.0018	0.0228	6.4
6.5	0.0087	0.0116	0.0061	0.0149	0.0038	0.0184	0.0018	0.0223	6.5
6.6	0.0085	0.0114	0.0060	0.0145	0.0037	0.0180	0.0017	0.0218	6.6
6.7	0.0084	0.0112	0.0059	0.0142	0.0037	0.0176	0.0017	0.0213	6.7
6.8	0.0082	0.0109	0.0058	0.0140	0.0036	0.0173	0.0017	0.0209	6.8
6.9	0.0080	0.0107	0.0057	0.0137	0.0035	0.0169	0.0016	0.0205	6.9
7.0	0.0079	0.0105	0.0056	0.0134	0.0035	0.0166	0.0016	0.0201	7.0
7.1	0.0077	0.0103	0.0054	0.0131	0.0034	0.0163	0.0016	0.0197	7.1
7.2	0.0076	0.0101	0.0054	0.0129	0.0033	0.0159	0.0016	0.0193	7.2
7.3	0.0075	0.0099	0.0053	0.0126	0.0033	0.0156	0.0015	0.0189	7.3
7.4	0.0073	0.0097	0.0052	0.0124	0.0032	0.0153	0.0015	0.0185	7.4
7.5	0.0072	0.0096	0.0051	0.0122	0.0032	0.0150	0.0015	0.0182	7.5
7.6	0.0071	0.0094	0.0050	0.0120	0.0031	0.0148	0.0015	0.0178	7.6
7.7	0.0069	0.0092	0.0049	0.0117	0.0031	0.0145	0.0014	0.0175	7.7
7.8	0.0068	0.0091	0.0048	0.0115	0.0030	0.0142	0.0014	0.0172	7.8
7.9	0.0067	0.0089	0.0047	0.0113	0.0030	0.0140	0.0014	0.0169	7.9
8.0	0.0066	0.0087	0.0047	0.0111	0.0029	0.0137	0.0014	0.0166	8.0
8.1	0.0065	0.0086	0.0046	0.0109	0.0029	0.0135	0.0014	0.0163	8.1
8.2	0.0064	0.0085	0.0045	0.0107	0.0028	0.0132	0.0013	0.0160	8.2
8.3	0.0063	0.0083	0.0044	0.0106	0.0028	0.0130	0.0013	0.0157	8.3
8.4	0.0062	0.0082	0.0044	0.0104	0.0027	0.0128	0.0013	0.0154	8.4
8.5	0.0061	0.0080	0.0043	0.0102	0.0027	0.0126	0.0013	0.0152	8.5
8.6	0.0060	0.0079	0.0042	0.0101	0.0027	0.0124	0.0012	0.0149	8.6
8.7	0.0059	0.0078	0.0042	0.0099	0.0026	0.0122	0.0012	0.0147	8.7
8.8	0.0058	0.0077	0.0041	0.0097	0.0026	0.0120	0.0012	0.0144	8.8
8.9	0.0057	0.0076	0.0040	0.0096	0.0025	0.0118	0.0012	0.0142	8.9
9.0	0.0056	0.0074	0.0040	0.0094	0.0025	0.0116	0.0012	0.0140	9.0
9.1	0.0055	0.0073	0.0039	0.0093	0.0025	0.0114	0.0012	0.0138	9.1
9.2	0.0054	0.0072	0.0039	0.0091	0.0024	0.0113	0.0011	0.0135	9.2
9.3	0.0054	0.0071	0.0038	0.0090	0.0024	0.0111	0.0011	0.0133	9.3
9.4	0.0053	0.0070	0.0038	0.0089	0.0024	0.0109	0.0011	0.0131	9.4
9.5	0.0052	0.0069	0.0037	0.0087	0.0023	0.0107	0.0011	0.0129	9.5

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
9.6	0.0051	0.0068	0.0036	0.0086	0.0023	0.0106	0.0011	0.0127	9.6
9.7	0.0051	0.0067	0.0036	0.0085	0.0023	0.0104	0.0011	0.0125	9.7
9.8	0.0050	0.0066	0.0036	0.0084	0.0022	0.0103	0.0010	0.0124	9.8
9.9	0.0049	0.0065	0.0035	0.0082	0.0022	0.0101	0.0010	0.0122	9.9
10.0	0.0049	0.0064	0.0034	0.0081	0.0022	0.0100	0.0010	0.0120	10.0
11.	0.0043	0.0056	0.0030	0.0071	0.0019	0.0087	0.0009	0.0104	11.
12.	0.0038	0.0050	0.0027	0.0063	0.0017	0.0077	0.0008	0.0092	12.
13.	0.0034	0.0045	0.0024	0.0056	0.0015	0.0068	0.0007	0.0082	13.
14.	0.0031	0.0040	0.0022	0.0050	0.0014	0.0062	0.0007	0.0073	14.
15.	0.0028	0.0036	0.0020	0.0046	0.0013	0.0056	0.0006	0.0066	15.
16.	0.0025	0.0033	0.0018	0.0042	0.0012	0.0051	0.0005	0.0060	16.
17.	0.0023	0.0030	0.0017	0.0038	0.0011	0.0046	0.0005	0.0055	17.
18.	0.0021	0.0028	0.0015	0.0035	0.0010	0.0043	0.0005	0.0051	18.
19.	0.0020	0.0026	0.0014	0.0032	0.0009	0.0039	0.0004	0.0047	19.
20.	0.0018	0.0024	0.0013	0.0030	0.0008	0.0037	0.0004	0.0044	20.
21.	0.0017	0.0023	0.0012	0.0028	0.0008	0.0034	0.0004	0.0041	21.
22.	0.0016	0.0021	0.0012	0.0026	0.0007	0.0032	0.0004	0.0038	22.
23.	0.0015	0.0020	0.0011	0.0025	0.0007	0.0030	0.0003	0.0035	23.
24.	0.0014	0.0019	0.0010	0.0023	0.0007	0.0028	0.0003	0.0033	24.
25.	0.0014	0.0017	0.0010	0.0022	0.0006	0.0026	0.0003	0.0031	25.
26.	0.0013	0.0017	0.0009	0.0021	0.0006	0.0025	0.0003	0.0029	26.
27.	0.0012	0.0016	0.0009	0.0019	0.0006	0.0024	0.0003	0.0028	27.
28.	0.0011	0.0015	0.0008	0.0018	0.0005	0.0022	0.0003	0.0026	28.
29.	0.0011	0.0014	0.0008	0.0018	0.0005	0.0021	0.0002	0.0025	29.
30.	0.0010	0.0014	0.0007	0.0017	0.0005	0.0020	0.0002	0.0024	30.
31.	0.0010	0.0013	0.0007	0.0016	0.0005	0.0019	0.0002	0.0023	31.
32.	0.0009	0.0012	0.0007	0.0015	0.0004	0.0018	0.0002	0.0022	32.
33.	0.0009	0.0012	0.0007	0.0015	0.0004	0.0017	0.0002	0.0021	33.
34.	0.0009	0.0011	0.0006	0.0014	0.0004	0.0017	0.0002	0.0020	34.
35.	0.0008	0.0011	0.0006	0.0013	0.0004	0.0016	0.0002	0.0019	35.
36.	0.0008	0.0010	0.0006	0.0013	0.0004	0.0015	0.0002	0.0018	36.
37.	0.0008	0.0010	0.0006	0.0012	0.0004	0.0015	0.0002	0.0017	37.
38.	0.0007	0.0010	0.0005	0.0012	0.0003	0.0014	0.0002	0.0017	38.
39.	0.0007	0.0009	0.0005	0.0011	0.0003	0.0014	0.0002	0.0016	39.
40.	0.0007	0.0009	0.0005	0.0011	0.0003	0.0013	0.0001	0.0016	40.
50.	0.0005	0.0006	0.0004	0.0008	0.0002	0.0009		0.0011	50.
60.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0007		0.0008	60.
70.	0.0003	0.0004	0.0002	0.0005	0.0001	0.0006		0.0007	70.
80.	0.0002	0.0003	0.0002	0.0004		0.0005		0.0005	80.
90.	0.0002	0.0003	0.0001	0.0003		0.0004		0.0005	90.
100.	0.0002	0.0002		0.0003		0.0003		0.0004	100.
110.	0.0002	0.0002		0.0002		0.0003		0.0003	110.
120.	0.0001	0.0002		0.0002		0.0002		0.0003	120.
130.		0.0001		0.0002		0.0002		0.0003	130.
140.				0.0002		0.0002		0.0002	140.
150.				0.0001		0.0002		0.0002	150.
160.						0.0002		0.0002	160.
170.						0.0001		0.0002	170.
180.								0.0002	180.
190.								0.0001	190.

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.00	0.3790	0.2308	0.2308	0.0808	0.0808	0.0214	0.0214	0.0000	0.00
0.01	0.3789	0.2350	0.2267	0.0822	0.0795	0.0216	0.0211	0.0000	0.01
0.02	0.3785	0.2393	0.2226	0.0837	0.0781	0.0219	0.0208	0.0000	0.02
0.03	0.3778	0.2437	0.2186	0.0852	0.0769	0.0222	0.0206	0.0000	0.03
0.04	0.3770	0.2481	0.2147	0.0867	0.0756	0.0225	0.0203	0.0000	0.04
0.05	0.3758	0.2527	0.2109	0.0882	0.0744	0.0228	0.0201	0.0000	0.05
0.06	0.3745	0.2572	0.2071	0.0899	0.0732	0.0231	0.0199	0.0000	0.06
0.07	0.3729	0.2619	0.2034	0.0915	0.0721	0.0234	0.0196	0.0000	0.07
0.08	0.3711	0.2665	0.1999	0.0933	0.0709	0.0237	0.0194	0.0000	0.08
0.09	0.3691	0.2712	0.1963	0.0950	0.0698	0.0241	0.0192	0.0000	0.09
0.10	0.3669	0.2760	0.1929	0.0969	0.0688	0.0244	0.0189	0.0000	0.10
0.11	0.3645	0.2808	0.1896	0.0988	0.0677	0.0248	0.0187	0.0000	0.11
0.12	0.3619	0.2856	0.1863	0.1007	0.0667	0.0251	0.0185	0.0000	0.12
0.13	0.3592	0.2903	0.1831	0.1027	0.0657	0.0255	0.0183	0.0000	0.13
0.14	0.3564	0.2951	0.1800	0.1048	0.0648	0.0258	0.0181	0.0000	0.14
0.15	0.3533	0.2999	0.1769	0.1070	0.0638	0.0262	0.0179	0.0000	0.15
0.16	0.3502	0.3046	0.1739	0.1092	0.0629	0.0266	0.0177	0.0000	0.16
0.17	0.3470	0.3092	0.1710	0.1115	0.0620	0.0270	0.0175	0.0000	0.17
0.18	0.3437	0.3138	0.1682	0.1139	0.0611	0.0275	0.0173	0.0000	0.18
0.19	0.3402	0.3183	0.1654	0.1164	0.0603	0.0279	0.0171	0.0000	0.19
0.20	0.3367	0.3227	0.1627	0.1189	0.0594	0.0283	0.0170	0.0000	0.20
0.21	0.3331	0.3270	0.1600	0.1216	0.0586	0.0288	0.0168	0.0000	0.21
0.22	0.3295	0.3312	0.1574	0.1243	0.0578	0.0292	0.0166	0.0000	0.22
0.23	0.3258	0.3352	0.1549	0.1272	0.0571	0.0297	0.0164	0.0000	0.23
0.24	0.3221	0.3391	0.1524	0.1301	0.0563	0.0302	0.0163	0.0000	0.24
0.25	0.3184	0.3428	0.1500	0.1332	0.0556	0.0307	0.0161	0.0000	0.25
0.26	0.3146	0.3463	0.1477	0.1363	0.0548	0.0313	0.0159	0.0000	0.26
0.27	0.3108	0.3496	0.1454	0.1396	0.0541	0.0318	0.0158	0.0000	0.27
0.28	0.3070	0.3527	0.1431	0.1430	0.0534	0.0324	0.0156	0.0000	0.28
0.29	0.3032	0.3556	0.1409	0.1465	0.0527	0.0329	0.0155	0.0000	0.29
0.30	0.2995	0.3583	0.1388	0.1502	0.0521	0.0335	0.0153	0.0000	0.30
0.31	0.2957	0.3607	0.1367	0.1539	0.0514	0.0342	0.0151	0.0000	0.31
0.32	0.2919	0.3629	0.1346	0.1578	0.0508	0.0348	0.0150	0.0000	0.32
0.33	0.2882	0.3649	0.1326	0.1619	0.0502	0.0355	0.0149	0.0000	0.33
0.34	0.2844	0.3666	0.1306	0.1661	0.0495	0.0361	0.0147	0.0000	0.34
0.35	0.2807	0.3681	0.1287	0.1704	0.0489	0.0369	0.0146	0.0000	0.35
0.36	0.2770	0.3693	0.1269	0.1748	0.0483	0.0376	0.0144	0.0000	0.36
0.37	0.2734	0.3703	0.1250	0.1794	0.0478	0.0384	0.0143	0.0000	0.37
0.38	0.2698	0.3711	0.1232	0.1841	0.0472	0.0392	0.0142	0.0000	0.38
0.39	0.2662	0.3717	0.1215	0.1889	0.0466	0.0400	0.0140	0.0000	0.39
0.40	0.2627	0.3720	0.1197	0.1939	0.0461	0.0409	0.0139	0.0000	0.40
0.41	0.2591	0.3721	0.1181	0.1989	0.0456	0.0418	0.0138	0.0000	0.41
0.42	0.2557	0.3720	0.1164	0.2041	0.0450	0.0427	0.0137	0.0000	0.42
0.43	0.2523	0.3717	0.1148	0.2094	0.0445	0.0437	0.0135	0.0000	0.43
0.44	0.2489	0.3712	0.1132	0.2147	0.0440	0.0447	0.0134	0.0000	0.44
0.45	0.2456	0.3706	0.1117	0.2201	0.0435	0.0458	0.0133	0.0000	0.45
0.46	0.2423	0.3697	0.1102	0.2256	0.0430	0.0469	0.0132	0.0000	0.46
0.47	0.2390	0.3687	0.1087	0.2311	0.0426	0.0481	0.0131	0.0000	0.47
0.48	0.2358	0.3675	0.1073	0.2367	0.0421	0.0493	0.0129	0.0000	0.48
0.49	0.2327	0.3661	0.1058	0.2423	0.0416	0.0506	0.0128	0.0000	0.49
0.50	0.2296	0.3646	0.1045	0.2478	0.0412	0.0520	0.0127	0.0000	0.50

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
0.51	0.2265	0.3630	0.1031	0.2534	0.0407	0.0534	0.0126	0.0000	0.51
0.52	0.2235	0.3612	0.1018	0.2589	0.0403	0.0549	0.0125	0.0000	0.52
0.53	0.2205	0.3594	0.1004	0.2643	0.0399	0.0566	0.0124	0.0000	0.53
0.54	0.2176	0.3574	0.0992	0.2697	0.0395	0.0583	0.0123	0.0000	0.54
0.55	0.2147	0.3553	0.0979	0.2750	0.0390	0.0601	0.0122	0.0000	0.55
0.56	0.2119	0.3531	0.0967	0.2802	0.0386	0.0620	0.0121	0.0000	0.56
0.57	0.2091	0.3508	0.0955	0.2853	0.0382	0.0640	0.0120	0.0000	0.57
0.58	0.2064	0.3484	0.0943	0.2902	0.0378	0.0662	0.0119	0.0000	0.58
0.59	0.2037	0.3460	0.0931	0.2951	0.0375	0.0685	0.0118	0.0000	0.59
0.60	0.2010	0.3435	0.0920	0.2997	0.0371	0.0709	0.0117	0.0000	0.60
0.61	0.1984	0.3409	0.0909	0.3043	0.0367	0.0735	0.0116	0.0000	0.61
0.62	0.1959	0.3383	0.0898	0.3086	0.0363	0.0763	0.0115	0.0000	0.62
0.63	0.1933	0.3356	0.0887	0.3128	0.0360	0.0792	0.0114	0.0000	0.63
0.64	0.1909	0.3329	0.0876	0.3168	0.0356	0.0823	0.0113	0.0000	0.64
0.65	0.1884	0.3302	0.0866	0.3205	0.0353	0.0856	0.0112	0.0000	0.65
0.66	0.1860	0.3274	0.0856	0.3242	0.0349	0.0890	0.0111	0.0000	0.66
0.67	0.1837	0.3245	0.0846	0.3276	0.0346	0.0927	0.0111	0.0000	0.67
0.68	0.1813	0.3217	0.0836	0.3308	0.0343	0.0965	0.0110	0.0000	0.68
0.69	0.1791	0.3188	0.0827	0.3337	0.0339	0.1005	0.0109	0.0000	0.69
0.70	0.1768	0.3159	0.0817	0.3365	0.0336	0.1046	0.0108	0.0000	0.70
0.71	0.1746	0.3130	0.0808	0.3391	0.0333	0.1090	0.0107	0.0000	0.71
0.72	0.1724	0.3101	0.0799	0.3415	0.0330	0.1136	0.0106	0.0000	0.72
0.73	0.1703	0.3072	0.0790	0.3437	0.0327	0.1183	0.0105	0.0000	0.73
0.74	0.1682	0.3043	0.0781	0.3457	0.0324	0.1232	0.0105	0.0000	0.74
0.75	0.1662	0.3014	0.0772	0.3475	0.0321	0.1282	0.0104	0.0000	0.75
0.76	0.1642	0.2985	0.0764	0.3491	0.0318	0.1334	0.0103	0.0000	0.76
0.77	0.1622	0.2955	0.0755	0.3505	0.0315	0.1387	0.0102	0.0001	0.77
0.78	0.1602	0.2926	0.0747	0.3517	0.0312	0.1441	0.0102	0.0002	0.78
0.79	0.1583	0.2897	0.0739	0.3527	0.0309	0.1496	0.0101	0.0002	0.79
0.80	0.1564	0.2868	0.0731	0.3536	0.0307	0.1553	0.0100	0.0004	0.80
0.81	0.1545	0.2839	0.0724	0.3543	0.0304	0.1610	0.0099	0.0005	0.81
0.82	0.1527	0.2811	0.0716	0.3548	0.0301	0.1668	0.0099	0.0007	0.82
0.83	0.1509	0.2782	0.0708	0.3552	0.0298	0.1726	0.0098	0.0009	0.83
0.84	0.1491	0.2754	0.0701	0.3554	0.0296	0.1784	0.0097	0.0012	0.84
0.85	0.1474	0.2726	0.0694	0.3555	0.0293	0.1843	0.0097	0.0015	0.85
0.86	0.1457	0.2698	0.0686	0.3554	0.0291	0.1902	0.0096	0.0020	0.86
0.87	0.1440	0.2670	0.0679	0.3552	0.0288	0.1961	0.0095	0.0025	0.87
0.88	0.1424	0.2642	0.0672	0.3548	0.0286	0.2019	0.0095	0.0031	0.88
0.89	0.1407	0.2615	0.0666	0.3543	0.0283	0.2077	0.0094	0.0038	0.89
0.90	0.1391	0.2588	0.0659	0.3537	0.0281	0.2135	0.0093	0.0047	0.90
0.91	0.1376	0.2561	0.0652	0.3530	0.0279	0.2192	0.0093	0.0057	0.91
0.92	0.1360	0.2534	0.0646	0.3521	0.0276	0.2248	0.0092	0.0069	0.92
0.93	0.1345	0.2508	0.0639	0.3512	0.0274	0.2304	0.0091	0.0082	0.93
0.94	0.1330	0.2481	0.0633	0.3502	0.0272	0.2358	0.0091	0.0096	0.94
0.95	0.1315	0.2455	0.0627	0.3490	0.0270	0.2412	0.0090	0.0113	0.95
0.96	0.1301	0.2430	0.0621	0.3478	0.0267	0.2464	0.0089	0.0131	0.96
0.97	0.1286	0.2404	0.0615	0.3465	0.0265	0.2515	0.0089	0.0152	0.97
0.98	0.1272	0.2379	0.0609	0.3451	0.0263	0.2564	0.0088	0.0174	0.98
0.99	0.1259	0.2354	0.0603	0.3436	0.0261	0.2613	0.0088	0.0199	0.99
1.00	0.1245	0.2330	0.0597	0.3421	0.0259	0.2660	0.0087	0.0226	1.00

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
1.01	0.1232	0.2306	0.0592	0.3405	0.0257	0.2705	0.0086	0.0254	1.01
1.02	0.1218	0.2281	0.0586	0.3388	0.0255	0.2749	0.0086	0.0285	1.02
1.03	0.1205	0.2258	0.0581	0.3371	0.0253	0.2791	0.0085	0.0318	1.03
1.04	0.1193	0.2234	0.0575	0.3353	0.0251	0.2832	0.0085	0.0353	1.04
1.05	0.1180	0.2211	0.0570	0.3335	0.0249	0.2871	0.0084	0.0391	1.05
1.06	0.1168	0.2188	0.0565	0.3316	0.0247	0.2909	0.0084	0.0430	1.06
1.07	0.1156	0.2165	0.0559	0.3297	0.0245	0.2944	0.0083	0.0471	1.07
1.08	0.1144	0.2143	0.0554	0.3277	0.0243	0.2979	0.0083	0.0514	1.08
1.09	0.1132	0.2120	0.0549	0.3257	0.0241	0.3011	0.0082	0.0559	1.09
1.10	0.1120	0.2098	0.0544	0.3237	0.0239	0.3042	0.0082	0.0605	1.10
1.11	0.1109	0.2077	0.0539	0.3216	0.0237	0.3071	0.0081	0.0653	1.11
1.12	0.1097	0.2055	0.0535	0.3196	0.0236	0.3099	0.0081	0.0703	1.12
1.13	0.1086	0.2034	0.0530	0.3174	0.0234	0.3125	0.0080	0.0754	1.13
1.14	0.1075	0.2014	0.0525	0.3153	0.0232	0.3149	0.0079	0.0806	1.14
1.15	0.1065	0.1993	0.0521	0.3131	0.0230	0.3172	0.0079	0.0860	1.15
1.16	0.1054	0.1973	0.0516	0.3110	0.0229	0.3193	0.0079	0.0914	1.16
1.17	0.1044	0.1953	0.0512	0.3088	0.0227	0.3213	0.0078	0.0970	1.17
1.18	0.1033	0.1933	0.0507	0.3066	0.0226	0.3231	0.0078	0.1026	1.18
1.19	0.1023	0.1913	0.0503	0.3043	0.0224	0.3248	0.0077	0.1083	1.19
1.20	0.1013	0.1894	0.0499	0.3021	0.0222	0.3263	0.0077	0.1140	1.20
1.21	0.1003	0.1875	0.0494	0.2999	0.0221	0.3277	0.0076	0.1198	1.21
1.22	0.0994	0.1856	0.0490	0.2976	0.0219	0.3289	0.0076	0.1256	1.22
1.23	0.0984	0.1837	0.0486	0.2954	0.0217	0.3300	0.0075	0.1315	1.23
1.24	0.0975	0.1819	0.0482	0.2931	0.0216	0.3310	0.0075	0.1373	1.24
1.25	0.0965	0.1801	0.0478	0.2909	0.0214	0.3319	0.0074	0.1431	1.25
1.26	0.0956	0.1783	0.0474	0.2886	0.0213	0.3326	0.0074	0.1489	1.26
1.27	0.0947	0.1765	0.0470	0.2864	0.0211	0.3332	0.0073	0.1547	1.27
1.28	0.0938	0.1748	0.0466	0.2841	0.0210	0.3337	0.0073	0.1605	1.28
1.29	0.0930	0.1731	0.0462	0.2818	0.0208	0.3341	0.0073	0.1662	1.29
1.30	0.0921	0.1714	0.0459	0.2796	0.0207	0.3343	0.0072	0.1718	1.30
1.31	0.0912	0.1697	0.0455	0.2774	0.0206	0.3345	0.0072	0.1774	1.31
1.32	0.0904	0.1680	0.0451	0.2751	0.0204	0.3345	0.0071	0.1829	1.32
1.33	0.0896	0.1664	0.0448	0.2729	0.0203	0.3345	0.0071	0.1884	1.33
1.34	0.0887	0.1648	0.0444	0.2707	0.0201	0.3344	0.0071	0.1937	1.34
1.35	0.0879	0.1632	0.0440	0.2685	0.0200	0.3342	0.0070	0.1990	1.35
1.36	0.0871	0.1616	0.0437	0.2663	0.0199	0.3338	0.0070	0.2041	1.36
1.37	0.0864	0.1601	0.0433	0.2641	0.0197	0.3334	0.0069	0.2092	1.37
1.38	0.0856	0.1586	0.0430	0.2619	0.0196	0.3330	0.0069	0.2141	1.38
1.39	0.0848	0.1571	0.0427	0.2597	0.0195	0.3324	0.0068	0.2190	1.39
1.40	0.0841	0.1556	0.0423	0.2576	0.0193	0.3318	0.0068	0.2237	1.40
1.41	0.0833	0.1541	0.0420	0.2554	0.0192	0.3311	0.0068	0.2283	1.41
1.42	0.0826	0.1526	0.0417	0.2533	0.0191	0.3303	0.0067	0.2328	1.42
1.43	0.0819	0.1512	0.0414	0.2512	0.0190	0.3295	0.0067	0.2371	1.43
1.44	0.0812	0.1498	0.0410	0.2491	0.0188	0.3286	0.0067	0.2414	1.44
1.45	0.0804	0.1484	0.0407	0.2470	0.0187	0.3276	0.0066	0.2454	1.45
1.46	0.0798	0.1470	0.0404	0.2449	0.0186	0.3266	0.0066	0.2494	1.46
1.47	0.0791	0.1456	0.0401	0.2428	0.0185	0.3256	0.0065	0.2533	1.47
1.48	0.0784	0.1443	0.0398	0.2408	0.0183	0.3244	0.0065	0.2570	1.48
1.49	0.0777	0.1430	0.0395	0.2388	0.0182	0.3233	0.0065	0.2605	1.49
1.50	0.0771	0.1417	0.0392	0.2367	0.0181	0.3221	0.0064	0.2640	1.50

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
1.51	0.0764	0.1404	0.0390	0.2347	0.0180	0.3208	0.0064	0.2673	1.51
1.52	0.0758	0.1391	0.0387	0.2327	0.0179	0.3195	0.0064	0.2705	1.52
1.53	0.0752	0.1378	0.0384	0.2308	0.0178	0.3182	0.0063	0.2735	1.53
1.54	0.0745	0.1366	0.0381	0.2288	0.0177	0.3168	0.0063	0.2764	1.54
1.55	0.0739	0.1354	0.0378	0.2269	0.0176	0.3154	0.0063	0.2792	1.55
1.56	0.0733	0.1342	0.0376	0.2250	0.0174	0.3140	0.0062	0.2819	1.56
1.57	0.0727	0.1330	0.0373	0.2231	0.0173	0.3125	0.0062	0.2844	1.57
1.58	0.0721	0.1318	0.0370	0.2212	0.0172	0.3110	0.0062	0.2868	1.58
1.59	0.0715	0.1306	0.0368	0.2193	0.0171	0.3095	0.0061	0.2891	1.59
1.60	0.0709	0.1295	0.0365	0.2175	0.0170	0.3080	0.0061	0.2913	1.60
1.61	0.0704	0.1283	0.0362	0.2156	0.0169	0.3064	0.0061	0.2933	1.61
1.62	0.0698	0.1272	0.0360	0.2138	0.0168	0.3048	0.0060	0.2953	1.62
1.63	0.0692	0.1261	0.0357	0.2120	0.0167	0.3032	0.0060	0.2971	1.63
1.64	0.0687	0.1250	0.0355	0.2102	0.0166	0.3016	0.0060	0.2988	1.64
1.65	0.0681	0.1239	0.0352	0.2084	0.0165	0.2999	0.0059	0.3004	1.65
1.66	0.0676	0.1228	0.0350	0.2067	0.0164	0.2983	0.0059	0.3019	1.66
1.67	0.0671	0.1218	0.0347	0.2049	0.0163	0.2966	0.0059	0.3032	1.67
1.68	0.0665	0.1207	0.0345	0.2032	0.0162	0.2949	0.0059	0.3045	1.68
1.69	0.0660	0.1197	0.0343	0.2015	0.0161	0.2932	0.0058	0.3057	1.69
1.70	0.0655	0.1187	0.0340	0.1998	0.0160	0.2915	0.0058	0.3067	1.70
1.71	0.0650	0.1177	0.0338	0.1981	0.0159	0.2898	0.0058	0.3077	1.71
1.72	0.0645	0.1167	0.0336	0.1965	0.0158	0.2880	0.0057	0.3086	1.72
1.73	0.0640	0.1157	0.0333	0.1948	0.0157	0.2863	0.0057	0.3094	1.73
1.74	0.0635	0.1148	0.0331	0.1932	0.0156	0.2845	0.0057	0.3101	1.74
1.75	0.0630	0.1138	0.0329	0.1916	0.0156	0.2828	0.0056	0.3107	1.75
1.76	0.0626	0.1128	0.0327	0.1900	0.0155	0.2810	0.0056	0.3112	1.76
1.77	0.0621	0.1119	0.0325	0.1884	0.0154	0.2793	0.0056	0.3116	1.77
1.78	0.0616	0.1110	0.0323	0.1869	0.0153	0.2775	0.0056	0.3120	1.78
1.79	0.0612	0.1101	0.0320	0.1853	0.0152	0.2758	0.0055	0.3123	1.79
1.80	0.0607	0.1092	0.0318	0.1838	0.0151	0.2740	0.0055	0.3125	1.80
1.81	0.0603	0.1083	0.0316	0.1823	0.0150	0.2723	0.0055	0.3126	1.81
1.82	0.0598	0.1074	0.0314	0.1808	0.0149	0.2705	0.0055	0.3127	1.82
1.83	0.0594	0.1065	0.0312	0.1793	0.0149	0.2687	0.0054	0.3127	1.83
1.84	0.0589	0.1057	0.0310	0.1779	0.0148	0.2670	0.0054	0.3126	1.84
1.85	0.0585	0.1048	0.0308	0.1764	0.0147	0.2652	0.0054	0.3125	1.85
1.86	0.0581	0.1040	0.0306	0.1750	0.0146	0.2635	0.0054	0.3123	1.86
1.87	0.0576	0.1032	0.0304	0.1735	0.0145	0.2617	0.0053	0.3120	1.87
1.88	0.0572	0.1023	0.0302	0.1721	0.0145	0.2600	0.0053	0.3117	1.88
1.89	0.0568	0.1015	0.0300	0.1707	0.0144	0.2582	0.0053	0.3114	1.89
1.90	0.0564	0.1007	0.0298	0.1694	0.0143	0.2565	0.0053	0.3109	1.90
1.91	0.0560	0.0999	0.0297	0.1680	0.0142	0.2548	0.0052	0.3105	1.91
1.92	0.0556	0.0991	0.0295	0.1666	0.0141	0.2530	0.0052	0.3100	1.92
1.93	0.0552	0.0984	0.0293	0.1653	0.0141	0.2513	0.0052	0.3094	1.93
1.94	0.0548	0.0976	0.0291	0.1640	0.0140	0.2496	0.0052	0.3088	1.94
1.95	0.0544	0.0968	0.0289	0.1627	0.0139	0.2479	0.0051	0.3081	1.95
1.96	0.0541	0.0961	0.0287	0.1614	0.0138	0.2462	0.0051	0.3074	1.96
1.97	0.0537	0.0954	0.0286	0.1601	0.0138	0.2445	0.0051	0.3068	1.97
1.98	0.0533	0.0946	0.0284	0.1588	0.0137	0.2428	0.0051	0.3059	1.98
1.99	0.0530	0.0939	0.0282	0.1576	0.0136	0.2411	0.0050	0.3051	1.99
2.00	0.0526	0.0932	0.0280	0.1563	0.0135	0.2394	0.0050	0.3042	2.00

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
2.1	0.0491	0.0865	0.0264	0.1446	0.0129	0.2232	0.0048	0.2941	2.1
2.2	0.0460	0.0804	0.0249	0.1340	0.0122	0.2080	0.0046	0.2818	2.2
2.3	0.0432	0.0750	0.0236	0.1245	0.0116	0.1938	0.0044	0.2683	2.3
2.4	0.0407	0.0701	0.0224	0.1158	0.0111	0.1807	0.0042	0.2544	2.4
2.5	0.0383	0.0657	0.0212	0.1080	0.0106	0.1686	0.0040	0.2405	2.5
2.6	0.0362	0.0617	0.0202	0.1010	0.0101	0.1575	0.0039	0.2270	2.6
2.7	0.0343	0.0580	0.0192	0.0945	0.0097	0.1473	0.0037	0.2139	2.7
2.8	0.0325	0.0547	0.0183	0.0887	0.0093	0.1380	0.0036	0.2014	2.8
2.9	0.0309	0.0517	0.0175	0.0834	0.0089	0.1294	0.0035	0.1897	2.9
3.0	0.0293	0.0489	0.0167	0.0785	0.0086	0.1215	0.0033	0.1786	3.0
3.1	0.0279	0.0463	0.0160	0.0740	0.0082	0.1143	0.0032	0.1683	3.1
3.2	0.0266	0.0439	0.0153	0.0699	0.0079	0.1076	0.0031	0.1587	3.2
3.3	0.0254	0.0417	0.0147	0.0661	0.0076	0.1015	0.0030	0.1497	3.3
3.4	0.0243	0.0397	0.0141	0.0626	0.0073	0.0959	0.0029	0.1413	3.4
3.5	0.0232	0.0378	0.0135	0.0594	0.0071	0.0907	0.0028	0.1336	3.5
3.6	0.0223	0.0360	0.0130	0.0564	0.0068	0.0858	0.0027	0.1263	3.6
3.7	0.0213	0.0344	0.0125	0.0537	0.0066	0.0814	0.0027	0.1196	3.7
3.8	0.0205	0.0329	0.0121	0.0511	0.0064	0.0773	0.0026	0.1134	3.8
3.9	0.0197	0.0315	0.0116	0.0487	0.0062	0.0734	0.0025	0.1076	3.9
4.0	0.0189	0.0302	0.0112	0.0465	0.0060	0.0699	0.0024	0.1022	4.0
4.1	0.0182	0.0289	0.0108	0.0445	0.0058	0.0665	0.0023	0.0971	4.1
4.2	0.0175	0.0278	0.0105	0.0425	0.0056	0.0635	0.0023	0.0924	4.2
4.3	0.0169	0.0267	0.0101	0.0407	0.0054	0.0606	0.0022	0.0880	4.3
4.4	0.0163	0.0257	0.0098	0.0390	0.0053	0.0579	0.0022	0.0839	4.4
4.5	0.0157	0.0247	0.0095	0.0374	0.0051	0.0554	0.0021	0.0801	4.5
4.6	0.0152	0.0238	0.0092	0.0359	0.0050	0.0530	0.0020	0.0765	4.6
4.7	0.0147	0.0229	0.0089	0.0345	0.0049	0.0508	0.0020	0.0731	4.7
4.8	0.0142	0.0221	0.0086	0.0332	0.0047	0.0487	0.0019	0.0700	4.8
4.9	0.0138	0.0213	0.0084	0.0319	0.0046	0.0467	0.0019	0.0670	4.9
5.0	0.0133	0.0206	0.0081	0.0308	0.0045	0.0449	0.0018	0.0642	5.0
5.1	0.0129	0.0199	0.0079	0.0297	0.0044	0.0432	0.0018	0.0616	5.1
5.2	0.0125	0.0192	0.0077	0.0286	0.0042	0.0415	0.0018	0.0591	5.2
5.3	0.0121	0.0186	0.0075	0.0276	0.0041	0.0400	0.0017	0.0568	5.3
5.4	0.0118	0.0180	0.0073	0.0267	0.0040	0.0385	0.0017	0.0546	5.4
5.5	0.0114	0.0175	0.0071	0.0258	0.0039	0.0371	0.0016	0.0525	5.5
5.6	0.0111	0.0169	0.0069	0.0249	0.0038	0.0358	0.0016	0.0506	5.6
5.7	0.0108	0.0164	0.0067	0.0241	0.0037	0.0346	0.0016	0.0487	5.7
5.8	0.0105	0.0159	0.0065	0.0233	0.0036	0.0334	0.0015	0.0469	5.8
5.9	0.0102	0.0155	0.0064	0.0226	0.0036	0.0323	0.0015	0.0453	5.9
6.0	0.0099	0.0150	0.0062	0.0219	0.0035	0.0312	0.0015	0.0437	6.0
6.1	0.0097	0.0146	0.0061	0.0213	0.0034	0.0302	0.0014	0.0422	6.1
6.2	0.0094	0.0142	0.0059	0.0206	0.0033	0.0293	0.0014	0.0408	6.2
6.3	0.0092	0.0138	0.0058	0.0200	0.0032	0.0283	0.0014	0.0394	6.3
6.4	0.0090	0.0134	0.0056	0.0194	0.0032	0.0275	0.0014	0.0381	6.4
6.5	0.0087	0.0131	0.0055	0.0189	0.0031	0.0266	0.0013	0.0369	6.5
6.6	0.0085	0.0127	0.0054	0.0184	0.0030	0.0258	0.0013	0.0357	6.6
6.7	0.0083	0.0124	0.0053	0.0179	0.0030	0.0251	0.0013	0.0346	6.7
6.8	0.0081	0.0121	0.0051	0.0174	0.0029	0.0244	0.0012	0.0336	6.8
6.9	0.0079	0.0118	0.0050	0.0169	0.0029	0.0237	0.0012	0.0325	6.9
7.0	0.0077	0.0115	0.0049	0.0165	0.0028	0.0230	0.0012	0.0316	7.0

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
7.1	0.0076	0.0112	0.0048	0.0160	0.0027	0.0224	0.0012	0.0307	7.1
7.2	0.0074	0.0109	0.0047	0.0156	0.0027	0.0217	0.0011	0.0298	7.2
7.3	0.0072	0.0107	0.0046	0.0152	0.0026	0.0212	0.0011	0.0289	7.3
7.4	0.0071	0.0104	0.0045	0.0148	0.0026	0.0206	0.0011	0.0281	7.4
7.5	0.0069	0.0102	0.0044	0.0145	0.0025	0.0201	0.0011	0.0273	7.5
7.6	0.0068	0.0100	0.0044	0.0141	0.0025	0.0195	0.0011	0.0266	7.6
7.7	0.0066	0.0097	0.0043	0.0138	0.0024	0.0190	0.0010	0.0259	7.7
7.8	0.0065	0.0095	0.0042	0.0134	0.0024	0.0186	0.0010	0.0252	7.8
7.9	0.0064	0.0093	0.0041	0.0131	0.0023	0.0181	0.0010	0.0245	7.9
8.0	0.0062	0.0091	0.0040	0.0128	0.0023	0.0177	0.0010	0.0239	8.0
8.1	0.0061	0.0089	0.0039	0.0125	0.0023	0.0172	0.0010	0.0233	8.1
8.2	0.0060	0.0087	0.0039	0.0123	0.0022	0.0168	0.0010	0.0227	8.2
8.3	0.0059	0.0085	0.0038	0.0120	0.0022	0.0164	0.0009	0.0221	8.3
8.4	0.0058	0.0084	0.0037	0.0117	0.0021	0.0160	0.0009	0.0216	8.4
8.5	0.0056	0.0082	0.0036	0.0115	0.0021	0.0157	0.0009	0.0210	8.5
8.6	0.0055	0.0080	0.0036	0.0112	0.0021	0.0153	0.0009	0.0205	8.6
8.7	0.0054	0.0079	0.0035	0.0110	0.0020	0.0150	0.0009	0.0201	8.7
8.8	0.0053	0.0077	0.0035	0.0107	0.0020	0.0146	0.0009	0.0196	8.8
8.9	0.0052	0.0076	0.0034	0.0105	0.0020	0.0143	0.0009	0.0191	8.9
9.0	0.0051	0.0074	0.0033	0.0103	0.0019	0.0140	0.0008	0.0187	9.0
9.1	0.0050	0.0073	0.0033	0.0101	0.0019	0.0137	0.0008	0.0183	9.1
9.2	0.0050	0.0071	0.0032	0.0099	0.0019	0.0134	0.0008	0.0179	9.2
9.3	0.0049	0.0070	0.0032	0.0097	0.0018	0.0131	0.0008	0.0175	9.3
9.4	0.0048	0.0069	0.0031	0.0095	0.0018	0.0129	0.0008	0.0171	9.4
9.5	0.0047	0.0067	0.0031	0.0093	0.0018	0.0126	0.0008	0.0167	9.5
9.6	0.0046	0.0066	0.0030	0.0092	0.0018	0.0123	0.0008	0.0164	9.6
9.7	0.0045	0.0065	0.0030	0.0090	0.0017	0.0121	0.0008	0.0160	9.7
9.8	0.0045	0.0064	0.0029	0.0088	0.0017	0.0119	0.0007	0.0157	9.8
9.9	0.0044	0.0063	0.0029	0.0086	0.0017	0.0116	0.0007	0.0154	9.9
10.0	0.0043	0.0062	0.0028	0.0085	0.0017	0.0114	0.0007	0.0150	10.0
10.1	0.0042	0.0060	0.0028	0.0083	0.0016	0.0112	0.0007	0.0147	10.1
10.2	0.0042	0.0059	0.0027	0.0082	0.0016	0.0110	0.0007	0.0144	10.2
10.3	0.0041	0.0059	0.0027	0.0080	0.0016	0.0108	0.0007	0.0142	10.3
10.4	0.0040	0.0058	0.0027	0.0079	0.0016	0.0106	0.0007	0.0139	10.4
10.5	0.0040	0.0056	0.0026	0.0077	0.0015	0.0104	0.0007	0.0136	10.5
10.6	0.0039	0.0056	0.0026	0.0076	0.0015	0.0102	0.0007	0.0133	10.6
10.7	0.0038	0.0055	0.0025	0.0075	0.0015	0.0100	0.0007	0.0131	10.7
10.8	0.0038	0.0054	0.0025	0.0074	0.0015	0.0098	0.0007	0.0129	10.8
10.9	0.0037	0.0053	0.0025	0.0072	0.0015	0.0096	0.0006	0.0126	10.9
11.0	0.0037	0.0052	0.0024	0.0071	0.0014	0.0095	0.0006	0.0124	11.0
11.1	0.0036	0.0051	0.0024	0.0070	0.0014	0.0093	0.0006	0.0122	11.1
11.2	0.0036	0.0050	0.0024	0.0069	0.0014	0.0091	0.0006	0.0119	11.2
11.3	0.0035	0.0050	0.0023	0.0068	0.0014	0.0090	0.0006	0.0117	11.3
11.4	0.0035	0.0049	0.0023	0.0067	0.0014	0.0088	0.0006	0.0115	11.4
11.5	0.0034	0.0048	0.0023	0.0065	0.0014	0.0087	0.0006	0.0113	11.5
11.6	0.0034	0.0047	0.0022	0.0064	0.0013	0.0085	0.0006	0.0111	11.6
11.7	0.0033	0.0047	0.0022	0.0063	0.0013	0.0084	0.0006	0.0109	11.7
11.8	0.0033	0.0046	0.0022	0.0063	0.0013	0.0083	0.0006	0.0107	11.8
11.9	0.0032	0.0045	0.0021	0.0062	0.0013	0.0081	0.0006	0.0105	11.9
12.0	0.0032	0.0045	0.0021	0.0061	0.0013	0.0080	0.0006	0.0104	12.0

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = .75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	X
12.1	0.0031	0.0044	0.0021	0.0060	0.0012	0.0079	0.0006	0.0102	12.1
12.2	0.0031	0.0043	0.0021	0.0059	0.0012	0.0077	0.0005	0.0100	12.2
12.3	0.0030	0.0043	0.0020	0.0058	0.0012	0.0076	0.0005	0.0099	12.3
12.4	0.0030	0.0042	0.0020	0.0057	0.0012	0.0075	0.0005	0.0097	12.4
12.5	0.0030	0.0042	0.0020	0.0056	0.0012	0.0074	0.0005	0.0095	12.5
12.6	0.0029	0.0041	0.0020	0.0055	0.0012	0.0073	0.0005	0.0094	12.6
12.7	0.0029	0.0040	0.0019	0.0055	0.0012	0.0072	0.0005	0.0093	12.7
12.8	0.0029	0.0040	0.0019	0.0054	0.0011	0.0071	0.0005	0.0091	12.8
12.9	0.0028	0.0039	0.0019	0.0053	0.0011	0.0070	0.0005	0.0090	12.9
13.0	0.0028	0.0039	0.0019	0.0052	0.0011	0.0069	0.0005	0.0088	13.0
14.	0.0024	0.0034	0.0017	0.0046	0.0010	0.0059	0.0005	0.0076	14.
15.	0.0022	0.0030	0.0015	0.0040	0.0009	0.0052	0.0004	0.0066	15.
16.	0.0020	0.0027	0.0013	0.0036	0.0008	0.0046	0.0004	0.0058	16.
17.	0.0018	0.0024	0.0012	0.0032	0.0007	0.0041	0.0003	0.0052	17.
18.	0.0016	0.0022	0.0011	0.0029	0.0007	0.0037	0.0003	0.0046	18.
19.	0.0015	0.0020	0.0010	0.0026	0.0006	0.0033	0.0003	0.0042	19.
20.	0.0013	0.0018	0.0009	0.0024	0.0006	0.0030	0.0003	0.0038	20.
21.	0.0012	0.0017	0.0008	0.0022	0.0005	0.0027	0.0002	0.0034	21.
22.	0.0011	0.0015	0.0008	0.0020	0.0005	0.0025	0.0002	0.0031	22.
23.	0.0010	0.0014	0.0007	0.0018	0.0005	0.0023	0.0002	0.0029	23.
24.	0.0010	0.0013	0.0007	0.0017	0.0004	0.0021	0.0002	0.0026	24.
25.	0.0009	0.0012	0.0006	0.0016	0.0004	0.0020	0.0002	0.0024	25.
26.	0.0008	0.0011	0.0006	0.0015	0.0004	0.0018	0.0002	0.0023	26.
27.	0.0008	0.0011	0.0006	0.0014	0.0003	0.0017	0.0002	0.0021	27.
28.	0.0007	0.0010	0.0005	0.0013	0.0003	0.0016	0.0001	0.0020	28.
29.	0.0007	0.0009	0.0005	0.0012	0.0003	0.0015		0.0018	29.
30.	0.0007	0.0009	0.0005	0.0011	0.0003	0.0014		0.0017	30.
31.	0.0006	0.0008	0.0004	0.0011	0.0003	0.0013		0.0016	31.
32.	0.0006	0.0008	0.0004	0.0010	0.0003	0.0013		0.0015	32.
33.	0.0006	0.0007	0.0004	0.0010	0.0002	0.0012		0.0014	33.
34.	0.0005	0.0007	0.0004	0.0009	0.0002	0.0011		0.0014	34.
35.	0.0005	0.0007	0.0004	0.0009	0.0002	0.0011		0.0013	35.
36.	0.0005	0.0006	0.0003	0.0008	0.0002	0.0010		0.0012	36.
37.	0.0005	0.0006	0.0003	0.0008	0.0002	0.0010		0.0012	37.
38.	0.0004	0.0006	0.0003	0.0007	0.0002	0.0009		0.0011	38.
39.	0.0004	0.0006	0.0003	0.0007	0.0002	0.0009		0.0010	39.
40.	0.0004	0.0005	0.0003	0.0007	0.0002	0.0008		0.0010	40.
41.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0008		0.0010	41.
42.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0008		0.0009	42.
43.	0.0004	0.0005	0.0002	0.0006	0.0002	0.0007		0.0009	43.
53.	0.0002	0.0003	0.0002	0.0004	0.0001	0.0005		0.0006	53.
63.	0.0002	0.0002	0.0001	0.0003		0.0004		0.0004	63.
73.	0.0001	0.0002		0.0002		0.0003		0.0003	73.
83.		0.0001		0.0002		0.0002		0.0003	83.
93.				0.0001		0.0002		0.0002	93.
103.						0.0001		0.0002	103.
113.								0.0001	113.

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.00

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
0.0	0.3183	0.3096	0.3096	0.2925	0.2925	0.2761	0.2761	0.2622	0.2622	0.0
0.1	0.3152	0.3147	0.2999	0.3011	0.2814	0.2849	0.2657	0.2702	0.2532	0.1
0.2	0.3061	0.3143	0.2867	0.3061	0.2685	0.2915	0.2543	0.2768	0.2434	0.2
0.3	0.2920	0.3077	0.2711	0.3069	0.2545	0.2954	0.2423	0.2814	0.2332	0.3
0.4	0.2744	0.2953	0.2543	0.3025	0.2400	0.2958	0.2301	0.2836	0.2228	0.4
0.5	0.2547	0.2776	0.2369	0.2926	0.2254	0.2921	0.2178	0.2830	0.2123	0.5
0.6	0.2340	0.2562	0.2197	0.2772	0.2111	0.2838	0.2057	0.2789	0.2020	0.6
0.7	0.2136	0.2327	0.2030	0.2568	0.1973	0.2705	0.1940	0.2710	0.1919	0.7
0.8	0.1941	0.2087	0.1872	0.2325	0.1841	0.2523	0.1827	0.2588	0.1820	0.8
0.9	0.1759	0.1855	0.1724	0.2061	0.1716	0.2297	0.1719	0.2424	0.1726	0.9
1.0	0.1591	0.1640	0.1587	0.1793	0.1599	0.2036	0.1617	0.2218	0.1635	1.0
1.1	0.1440	0.1447	0.1461	0.1537	0.1490	0.1754	0.1520	0.1974	0.1549	1.1
1.2	0.1305	0.1277	0.1346	0.1306	0.1389	0.1469	0.1430	0.1701	0.1467	1.2
1.3	0.1183	0.1130	0.1241	0.1107	0.1295	0.1197	0.1344	0.1411	0.1389	1.3
1.4	0.1075	0.1002	0.1145	0.0939	0.1208	0.0955	0.1265	0.1120	0.1315	1.4
1.5	0.0979	0.0892	0.1058	0.0801	0.1128	0.0751	0.1190	0.0845	0.1246	1.5
1.6	0.0894	0.0798	0.0979	0.0689	0.1054	0.0589	0.1121	0.0599	0.1180	1.6
1.7	0.0818	0.0716	0.0907	0.0597	0.0986	0.0465	0.1056	0.0396	0.1119	1.7
1.8	0.0751	0.0646	0.0842	0.0522	0.0923	0.0374	0.0996	0.0240	0.1061	1.8
1.9	0.0691	0.0585	0.0783	0.0460	0.0865	0.0307	0.0940	0.0132	0.1006	1.9
2.0	0.0637	0.0531	0.0729	0.0409	0.0812	0.0258	0.0887	0.0065	0.0955	2.0
2.1	0.0588	0.0485	0.0680	0.0365	0.0763	0.0220	0.0838	0.0028	0.0907	2.1
2.2	0.0545	0.0444	0.0636	0.0329	0.0718	0.0191	0.0793	0.0010	0.0862	2.2
2.3	0.0506	0.0408	0.0595	0.0298	0.0676	0.0168	0.0751	0.0003	0.0820	2.3
2.4	0.0471	0.0376	0.0557	0.0271	0.0637	0.0150	0.0711		0.0780	2.4
2.5	0.0439	0.0348	0.0523	0.0247	0.0602	0.0134	0.0674		0.0742	2.5
2.6	0.0410	0.0322	0.0492	0.0227	0.0568	0.0121	0.0640		0.0707	2.6
2.7	0.0384	0.0300	0.0463	0.0209	0.0538	0.0110	0.0608		0.0674	2.7
2.8	0.0360	0.0279	0.0437	0.0193	0.0509	0.0101	0.0578		0.0643	2.8
2.9	0.0338	0.0261	0.0412	0.0179	0.0483	0.0092	0.0550		0.0614	2.9
3.0	0.0318	0.0244	0.0390	0.0166	0.0458	0.0085	0.0523		0.0586	3.0
3.1	0.0300	0.0229	0.0369	0.0155	0.0435	0.0079	0.0499		0.0560	3.1
3.2	0.0283	0.0215	0.0349	0.0145	0.0414	0.0073	0.0476		0.0536	3.2
3.3	0.0268	0.0202	0.0331	0.0136	0.0394	0.0068	0.0454		0.0513	3.3
3.4	0.0253	0.0191	0.0315	0.0128	0.0375	0.0064	0.0434		0.0491	3.4
3.5	0.0240	0.0180	0.0299	0.0120	0.0358	0.0060	0.0415		0.0471	3.5
3.6	0.0228	0.0171	0.0285	0.0113	0.0341	0.0056	0.0397		0.0451	3.6
3.7	0.0217	0.0162	0.0271	0.0107	0.0326	0.0053	0.0380		0.0433	3.7
3.8	0.0206	0.0153	0.0259	0.0101	0.0312	0.0050	0.0364		0.0416	3.8
3.9	0.0196	0.0146	0.0247	0.0096	0.0298	0.0047	0.0349		0.0399	3.9
4.0	0.0187	0.0139	0.0236	0.0091	0.0285	0.0045	0.0335		0.0384	4.0
4.1	0.0179	0.0132	0.0226	0.0086	0.0273	0.0042	0.0321		0.0369	4.1
4.2	0.0171	0.0126	0.0216	0.0082	0.0262	0.0040	0.0309		0.0355	4.2
4.3	0.0163	0.0120	0.0207	0.0078	0.0252	0.0038	0.0297		0.0342	4.3
4.4	0.0156	0.0115	0.0199	0.0075	0.0242	0.0036	0.0285		0.0329	4.4
4.5	0.0150	0.0110	0.0191	0.0072	0.0232	0.0035	0.0275		0.0317	4.5
4.6	0.0144	0.0105	0.0183	0.0068	0.0223	0.0033	0.0264		0.0306	4.6
4.7	0.0138	0.0101	0.0176	0.0065	0.0215	0.0032	0.0255		0.0295	4.7
4.8	0.0132	0.0097	0.0169	0.0063	0.0207	0.0030	0.0246		0.0285	4.8
4.9	0.0127	0.0093	0.0163	0.0060	0.0199	0.0029	0.0237		0.0275	4.9
5.0	0.0122	0.0089	0.0157	0.0058	0.0192	0.0028	0.0228		0.0266	5.0

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PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.00

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
5.1	0.0118	0.0086	0.0151	0.0055	0.0185	0.0027	0.0221		0.0257	5.1
5.2	0.0113	0.0083	0.0146	0.0053	0.0179	0.0026	0.0213		0.0248	5.2
5.3	0.0109	0.0080	0.0140	0.0051	0.0173	0.0025	0.0206		0.0240	5.3
5.4	0.0105	0.0077	0.0136	0.0049	0.0167	0.0024	0.0199		0.0232	5.4
5.5	0.0102	0.0074	0.0131	0.0048	0.0161	0.0023	0.0193		0.0225	5.5
5.6	0.0098	0.0071	0.0127	0.0046	0.0156	0.0022	0.0186		0.0218	5.6
5.7	0.0095	0.0069	0.0122	0.0044	0.0151	0.0021	0.0181		0.0211	5.7
5.8	0.0092	0.0067	0.0118	0.0043	0.0146	0.0021	0.0175		0.0205	5.8
5.9	0.0089	0.0064	0.0115	0.0041	0.0141	0.0020	0.0170		0.0199	5.9
6.0	0.0086	0.0062	0.0111	0.0040	0.0137	0.0019	0.0164		0.0193	6.0
6.1	0.0083	0.0060	0.0107	0.0039	0.0133	0.0019	0.0159		0.0187	6.1
6.2	0.0081	0.0058	0.0104	0.0038	0.0129	0.0018	0.0155		0.0181	6.2
6.3	0.0078	0.0057	0.0101	0.0036	0.0125	0.0017	0.0150		0.0176	6.3
6.4	0.0076	0.0055	0.0098	0.0035	0.0121	0.0017	0.0146		0.0171	6.4
6.5	0.0074	0.0053	0.0095	0.0034	0.0118	0.0016	0.0142		0.0166	6.5
6.6	0.0071	0.0052	0.0092	0.0033	0.0114	0.0016	0.0138		0.0162	6.6
6.7	0.0069	0.0050	0.0090	0.0032	0.0111	0.0015	0.0134		0.0157	6.7
6.8	0.0067	0.0049	0.0087	0.0031	0.0108	0.0015	0.0130		0.0153	6.8
6.9	0.0065	0.0047	0.0085	0.0030	0.0105	0.0014	0.0127		0.0149	6.9
7.0	0.0064	0.0046	0.0082	0.0029	0.0102	0.0014	0.0123		0.0145	7.0
7.1	0.0062	0.0045	0.0080	0.0029	0.0099	0.0014	0.0120		0.0141	7.1
7.2	0.0060	0.0044	0.0078	0.0028	0.0097	0.0013	0.0117		0.0138	7.2
7.3	0.0059	0.0042	0.0076	0.0027	0.0094	0.0013	0.0114		0.0134	7.3
7.4	0.0057	0.0041	0.0074	0.0026	0.0092	0.0013	0.0111		0.0131	7.4
7.5	0.0056	0.0040	0.0072	0.0026	0.0089	0.0012	0.0108		0.0127	7.5
7.6	0.0054	0.0039	0.0070	0.0025	0.0087	0.0012	0.0105		0.0124	7.6
7.7	0.0053	0.0038	0.0068	0.0024	0.0085	0.0012	0.0103		0.0121	7.7
7.8	0.0051	0.0037	0.0067	0.0024	0.0083	0.0011	0.0100		0.0118	7.8
7.9	0.0050	0.0036	0.0065	0.0023	0.0081	0.0011	0.0098		0.0115	7.9
8.0	0.0049	0.0035	0.0063	0.0023	0.0079	0.0011	0.0095		0.0113	8.0
8.1	0.0048	0.0034	0.0062	0.0022	0.0077	0.0010	0.0093		0.0110	8.1
8.2	0.0047	0.0034	0.0060	0.0021	0.0075	0.0010	0.0091		0.0107	8.2
8.3	0.0045	0.0033	0.0059	0.0021	0.0073	0.0010	0.0089		0.0105	8.3
8.4	0.0045	0.0032	0.0058	0.0020	0.0072	0.0010	0.0087		0.0103	8.4
8.5	0.0044	0.0031	0.0056	0.0020	0.0070	0.0010	0.0085		0.0100	8.5
8.6	0.0042	0.0031	0.0055	0.0019	0.0069	0.0009	0.0083		0.0098	8.6
8.7	0.0041	0.0030	0.0054	0.0019	0.0067	0.0009	0.0081		0.0096	8.7
8.8	0.0041	0.0029	0.0053	0.0019	0.0066	0.0009	0.0079		0.0094	8.8
8.9	0.0040	0.0029	0.0051	0.0018	0.0064	0.0009	0.0077		0.0092	8.9
9.0	0.0039	0.0028	0.0050	0.0018	0.0063	0.0008	0.0076		0.0090	9.0
9.1	0.0038	0.0027	0.0049	0.0017	0.0061	0.0008	0.0074		0.0088	9.1
9.2	0.0037	0.0027	0.0048	0.0017	0.0060	0.0008	0.0073		0.0086	9.2
9.3	0.0036	0.0026	0.0047	0.0017	0.0059	0.0008	0.0071		0.0084	9.3
9.4	0.0036	0.0026	0.0046	0.0016	0.0058	0.0008	0.0070		0.0082	9.4
9.5	0.0035	0.0025	0.0045	0.0016	0.0056	0.0008	0.0068		0.0081	9.5
9.6	0.0034	0.0025	0.0044	0.0016	0.0055	0.0007	0.0067		0.0079	9.6
9.7	0.0034	0.0024	0.0043	0.0015	0.0054	0.0007	0.0065		0.0077	9.7
9.8	0.0033	0.0024	0.0043	0.0015	0.0053	0.0007	0.0064		0.0076	9.8
9.9	0.0032	0.0023	0.0042	0.0015	0.0052	0.0007	0.0063		0.0074	9.9
10.0	0.0032	0.0023	0.0041	0.0014	0.0051	0.0007	0.0062		0.0073	10.0

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.00

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
11.	0.0026	0.0019	0.0034	0.0012	0.0042	0.0006	0.0051		0.0060	11.
12.	0.0022	0.0016	0.0028	0.0010	0.0035	0.0005	0.0043		0.0051	12.
13.	0.0019	0.0014	0.0024	0.0009	0.0030	0.0004	0.0036		0.0043	13.
14.	0.0016	0.0012	0.0021	0.0007	0.0026	0.0004	0.0031		0.0037	14.
15.	0.0014	0.0010	0.0018	0.0006	0.0023	0.0003	0.0027		0.0032	15.
16.	0.0012	0.0009	0.0016	0.0006	0.0020	0.0003	0.0024		0.0028	16.
17.	0.0011	0.0008	0.0014	0.0005	0.0018	0.0002	0.0021		0.0025	17.
18.	0.0010	0.0007	0.0013	0.0005	0.0016	0.0002	0.0019		0.0022	18.
19.	0.0009	0.0006	0.0011	0.0004	0.0014	0.0002	0.0017		0.0020	19.
20.	0.0008	0.0006	0.0010	0.0004	0.0013	0.0002	0.0015		0.0018	20.
21.	0.0007	0.0005	0.0009	0.0003	0.0011	0.0002	0.0014		0.0016	21.
22.	0.0007	0.0005	0.0008	0.0003	0.0010	0.0001	0.0012		0.0015	22.
23.	0.0006	0.0004	0.0008	0.0003	0.0009		0.0011		0.0014	23.
24.	0.0005	0.0004	0.0007	0.0003	0.0009		0.0010		0.0012	24.
25.	0.0005	0.0004	0.0006	0.0002	0.0008		0.0010		0.0011	25.
26.	0.0005	0.0003	0.0006	0.0002	0.0007		0.0009		0.0010	26.
27.	0.0004	0.0003	0.0006	0.0002	0.0007		0.0008		0.0010	27.
28.	0.0004	0.0003	0.0005	0.0002	0.0006		0.0008		0.0009	28.
29.	0.0004	0.0003	0.0005	0.0002	0.0006		0.0007		0.0008	29.
30.	0.0003	0.0002	0.0005	0.0002	0.0005		0.0007		0.0008	30.
40.	0.0002	0.0001	0.0002	0.0001	0.0003		0.0004		0.0004	40.
50.	0.0001		0.0002		0.0002		0.0002		0.0003	50.
60.			0.0001		0.0001		0.0002		0.0002	60.
70.							0.0001		0.0001	70.

ALPHA = 1.25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
0.0	0.2965	0.2375	0.2375	0.1578	0.1578	0.1090	0.1090	0.0808	0.0808	0.0
0.1	0.2985	0.2237	0.2507	0.1472	0.1690	0.1022	0.1163	0.0763	0.0856	0.1
0.2	0.2901	0.2097	0.2629	0.1371	0.1807	0.0958	0.1241	0.0721	0.0907	0.2
0.3	0.2827	0.1957	0.2738	0.1277	0.1928	0.0899	0.1323	0.0681	0.0962	0.3
0.4	0.2727	0.1821	0.2828	0.1188	0.2052	0.0843	0.1410	0.0644	0.1020	0.4
0.5	0.2606	0.1689	0.2896	0.1105	0.2177	0.0791	0.1502	0.0609	0.1082	0.5
0.6	0.2469	0.1563	0.2937	0.1029	0.2301	0.0743	0.1599	0.0577	0.1147	0.6
0.7	0.2320	0.1444	0.2950	0.0957	0.2422	0.0699	0.1700	0.0547	0.1216	0.7
0.8	0.2166	0.1333	0.2933	0.0891	0.2537	0.0657	0.1805	0.0518	0.1290	0.8
0.9	0.2009	0.1229	0.2884	0.0830	0.2643	0.0618	0.1913	0.0491	0.1367	0.9
1.0	0.1854	0.1133	0.2806	0.0774	0.2736	0.0582	0.2023	0.0466	0.1448	1.0
1.1	0.1703	0.1044	0.2700	0.0722	0.2813	0.0549	0.2134	0.0443	0.1533	1.1
1.2	0.1560	0.0962	0.2570	0.0674	0.2869	0.0517	0.2245	0.0421	0.1621	1.2
1.3	0.1424	0.0887	0.2421	0.0630	0.2903	0.0488	0.2354	0.0400	0.1713	1.3
1.4	0.1298	0.0819	0.2259	0.0589	0.2910	0.0461	0.2459	0.0381	0.1808	1.4
1.5	0.1182	0.0757	0.2089	0.0552	0.2889	0.0436	0.2557	0.0362	0.1905	1.5
1.6	0.1075	0.0699	0.1915	0.0517	0.2838	0.0412	0.2646	0.0345	0.2004	1.6
1.7	0.0979	0.0647	0.1744	0.0485	0.2758	0.0391	0.2723	0.0329	0.2104	1.7
1.8	0.0891	0.0600	0.1579	0.0455	0.2650	0.0370	0.2784	0.0314	0.2203	1.8
1.9	0.0811	0.0557	0.1422	0.0428	0.2516	0.0351	0.2828	0.0300	0.2300	1.9
2.0	0.0740	0.0517	0.1276	0.0403	0.2361	0.0333	0.2850	0.0286	0.2394	2.0
2.1	0.0676	0.0481	0.1142	0.0379	0.2189	0.0316	0.2849	0.0274	0.2483	2.1
2.2	0.0618	0.0448	0.1021	0.0357	0.2007	0.0301	0.2821	0.0262	0.2565	2.2
2.3	0.0566	0.0418	0.0912	0.0337	0.1820	0.0286	0.2767	0.0250	0.2637	2.3
2.4	0.0519	0.0390	0.0815	0.0319	0.1633	0.0272	0.2684	0.0240	0.2697	2.4
2.5	0.0477	0.0365	0.0729	0.0301	0.1452	0.0260	0.2575	0.0230	0.2743	2.5

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
2.6	0.0440	0.0342	0.0653	0.0285	0.1281	0.0248	0.2440	0.0220	0.2772	2.6
2.7	0.0406	0.0321	0.0587	0.0270	0.1123	0.0236	0.2282	0.0211	0.2782	2.7
2.8	0.0375	0.0301	0.0528	0.0256	0.0980	0.0226	0.2106	0.0203	0.2770	2.8
2.9	0.0348	0.0283	0.0476	0.0243	0.0852	0.0215	0.1917	0.0195	0.2734	2.9
3.0	0.0323	0.0267	0.0431	0.0231	0.0740	0.0206	0.1720	0.0187	0.2674	3.0
3.1	0.0300	0.0251	0.0391	0.0220	0.0643	0.0197	0.1522	0.0180	0.2588	3.1
3.2	0.0280	0.0237	0.0356	0.0209	0.0559	0.0189	0.1327	0.0173	0.2476	3.2
3.3	0.0261	0.0224	0.0325	0.0199	0.0488	0.0181	0.1142	0.0166	0.2340	3.3
3.4	0.0244	0.0212	0.0298	0.0190	0.0427	0.0173	0.0971	0.0160	0.2182	3.4
3.5	0.0228	0.0201	0.0273	0.0181	0.0376	0.0166	0.0816	0.0154	0.2005	3.5
3.6	0.0214	0.0190	0.0251	0.0173	0.0333	0.0160	0.0679	0.0149	0.1813	3.6
3.7	0.0201	0.0180	0.0232	0.0165	0.0296	0.0153	0.0562	0.0143	0.1612	3.7
3.8	0.0189	0.0171	0.0215	0.0158	0.0264	0.0147	0.0463	0.0138	0.1407	3.8
3.9	0.0178	0.0163	0.0199	0.0151	0.0237	0.0142	0.0382	0.0133	0.1203	3.9
4.0	0.0168	0.0155	0.0185	0.0145	0.0213	0.0136	0.0315	0.0129	0.1007	4.0
4.1	0.0158	0.0148	0.0172	0.0139	0.0193	0.0131	0.0262	0.0124	0.0824	4.1
4.2	0.0150	0.0141	0.0161	0.0133	0.0176	0.0126	0.0219	0.0120	0.0657	4.2
4.3	0.0142	0.0134	0.0150	0.0128	0.0161	0.0122	0.0186	0.0116	0.0510	4.3
4.4	0.0134	0.0128	0.0141	0.0123	0.0147	0.0117	0.0159	0.0113	0.0385	4.4
4.5	0.0128	0.0123	0.0132	0.0118	0.0135	0.0113	0.0137	0.0109	0.0282	4.5
4.6	0.0121	0.0117	0.0124	0.0113	0.0125	0.0109	0.0120	0.0105	0.0200	4.6
4.7	0.0115	0.0112	0.0117	0.0109	0.0116	0.0105	0.0106	0.0102	0.0137	4.7
4.8	0.0110	0.0108	0.0110	0.0105	0.0107	0.0102	0.0094	0.0099	0.0090	4.8
4.9	0.0105	0.0103	0.0104	0.0101	0.0100	0.0098	0.0085	0.0096	0.0057	4.9
5.0	0.0100	0.0099	0.0099	0.0097	0.0093	0.0095	0.0077	0.0093	0.0035	5.0
5.1	0.0095	0.0095	0.0093	0.0094	0.0087	0.0092	0.0069	0.0090	0.0020	5.1
5.2	0.0091	0.0091	0.0089	0.0090	0.0082	0.0089	0.0063	0.0087	0.0011	5.2
5.3	0.0087	0.0088	0.0084	0.0087	0.0077	0.0086	0.0058	0.0085	0.0006	5.3
5.4	0.0083	0.0085	0.0080	0.0084	0.0072	0.0084	0.0054	0.0082	0.0003	5.4
5.5	0.0080	0.0081	0.0076	0.0081	0.0068	0.0081	0.0050	0.0080	0.0001	5.5
5.6	0.0077	0.0078	0.0072	0.0079	0.0064	0.0078	0.0046	0.0078		5.6
5.7	0.0073	0.0075	0.0069	0.0076	0.0061	0.0076	0.0043	0.0076		5.7
5.8	0.0071	0.0073	0.0066	0.0074	0.0057	0.0074	0.0040	0.0073		5.8
5.9	0.0068	0.0070	0.0063	0.0071	0.0054	0.0072	0.0037	0.0071		5.9
6.0	0.0065	0.0068	0.0060	0.0069	0.0052	0.0069	0.0035	0.0069		6.0
6.1	0.0063	0.0065	0.0058	0.0067	0.0049	0.0068	0.0033	0.0068		6.1
6.2	0.0060	0.0063	0.0055	0.0065	0.0047	0.0065	0.0031	0.0066		6.2
6.3	0.0058	0.0061	0.0053	0.0063	0.0044	0.0064	0.0029	0.0064		6.3
6.4	0.0056	0.0059	0.0051	0.0061	0.0042	0.0062	0.0028	0.0062		6.4
6.5	0.0054	0.0057	0.0049	0.0059	0.0040	0.0060	0.0026	0.0061		6.5
6.6	0.0052	0.0055	0.0047	0.0057	0.0039	0.0059	0.0025	0.0059		6.6
6.7	0.0050	0.0054	0.0045	0.0056	0.0037	0.0057	0.0024	0.0058		6.7
6.8	0.0049	0.0052	0.0043	0.0054	0.0035	0.0055	0.0022	0.0056		6.8
6.9	0.0047	0.0050	0.0042	0.0053	0.0034	0.0054	0.0021	0.0055		6.9
7.0	0.0045	0.0049	0.0040	0.0051	0.0032	0.0053	0.0020	0.0053		7.0
7.1	0.0044	0.0047	0.0039	0.0050	0.0031	0.0051	0.0019	0.0052		7.1
7.2	0.0042	0.0046	0.0038	0.0048	0.0030	0.0050	0.0018	0.0051		7.2
7.3	0.0041	0.0045	0.0036	0.0047	0.0029	0.0049	0.0018	0.0050		7.3
7.4	0.0040	0.0043	0.0035	0.0046	0.0028	0.0047	0.0017	0.0049		7.4
7.5	0.0039	0.0042	0.0034	0.0045	0.0027	0.0046	0.0016	0.0047		7.5

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.25

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
7.6	0.0038	0.0041	0.0033	0.0043	0.0026	0.0045	0.0015	0.0046		7.6
7.7	0.0036	0.0040	0.0032	0.0042	0.0025	0.0044	0.0015	0.0045		7.7
7.8	0.0035	0.0039	0.0031	0.0041	0.0024	0.0043	0.0014	0.0044		7.8
7.9	0.0034	0.0038	0.0030	0.0040	0.0023	0.0042	0.0014	0.0043		7.9
8.0	0.0033	0.0037	0.0029	0.0039	0.0022	0.0041	0.0013	0.0042		8.0
9.	0.0025	0.0028	0.0021	0.0031	0.0016	0.0032	0.0009	0.0034		9.
10.	0.0020	0.0023	0.0016	0.0025	0.0012	0.0026	0.0007	0.0028		10.
11.	0.0016	0.0018	0.0013	0.0020	0.0009	0.0022	0.0005	0.0023		11.
12.	0.0013	0.0015	0.0010	0.0017	0.0008	0.0018	0.0004	0.0020		12.
13.	0.0011	0.0013	0.0009	0.0014	0.0006	0.0015	0.0003	0.0017		13.
14.	0.0009	0.0011	0.0007	0.0012	0.0005	0.0013	0.0003	0.0014		14.
15.	0.0008	0.0009	0.0006	0.0010	0.0004	0.0011	0.0002	0.0012		15.
16.	0.0007	0.0008	0.0005	0.0009	0.0004	0.0010	0.0002	0.0011		16.
17.	0.0006	0.0007	0.0005	0.0008	0.0003	0.0009	0.0002	0.0010		17.
18.	0.0005	0.0006	0.0004	0.0007	0.0003	0.0008	0.0001	0.0009		18.
19.	0.0005	0.0005	0.0003	0.0006	0.0002	0.0007		0.0008		19.
20.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0006		0.0007		20.
21.	0.0004	0.0004	0.0003	0.0005	0.0002	0.0006		0.0006		21.
22.	0.0003	0.0004	0.0002	0.0005	0.0002	0.0005		0.0006		22.
23.	0.0003	0.0003	0.0002	0.0004	0.0001	0.0005		0.0005		23.
24.	0.0003	0.0003	0.0002	0.0004		0.0004		0.0005		24.
25.	0.0002	0.0003	0.0002	0.0003		0.0004		0.0004		25.
26.	0.0002	0.0003	0.0002	0.0003		0.0003		0.0004		26.
27.	0.0002	0.0002	0.0001	0.0003		0.0003		0.0004		27.
28.	0.0002	0.0002		0.0003		0.0003		0.0003		28.
29.	0.0002	0.0002		0.0002		0.0003		0.0003		29.
30.	0.0002	0.0001		0.0002		0.0003		0.0003		30.
31.	0.0001			0.0002		0.0002		0.0003		31.
38.				0.0001		0.0001		0.0002		38.
48.								0.0001		48.

ALPHA = 1.50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
0.0	0.2873	0.2778	0.2778	0.2541	0.2541	0.2252	0.2252	0.1975	0.1975	0.0
0.1	0.2863	0.2712	0.2828	0.2442	0.2630	0.2144	0.2356	0.1872	0.2078	0.1
0.2	0.2831	0.2629	0.2858	0.2334	0.2707	0.2034	0.2455	0.1770	0.2180	0.2
0.3	0.2780	0.2533	0.2868	0.2221	0.2769	0.1923	0.2546	0.1669	0.2280	0.3
0.4	0.2710	0.2426	0.2857	0.2104	0.2815	0.1812	0.2627	0.1572	0.2375	0.4
0.5	0.2623	0.2310	0.2824	0.1986	0.2843	0.1704	0.2697	0.1477	0.2465	0.5
0.6	0.2521	0.2189	0.2772	0.1867	0.2851	0.1598	0.2754	0.1386	0.2548	0.6
0.7	0.2408	0.2063	0.2700	0.1749	0.2839	0.1495	0.2795	0.1299	0.2621	0.7
0.8	0.2285	0.1936	0.2610	0.1634	0.2806	0.1397	0.2819	0.1216	0.2683	0.8
0.9	0.2155	0.1808	0.2505	0.1522	0.2753	0.1303	0.2825	0.1137	0.2733	0.9
1.0	0.2020	0.1683	0.2386	0.1415	0.2680	0.1214	0.2811	0.1062	0.2769	1.0

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
1.1	0.1884	0.1561	0.2257	0.1313	0.2589	0.1129	0.2777	0.0992	0.2788	1.1
1.2	0.1748	0.1443	0.2119	0.1216	0.2481	0.1050	0.2724	0.0926	0.2790	1.2
1.3	0.1615	0.1331	0.1977	0.1124	0.2359	0.0975	0.2650	0.0865	0.2774	1.3
1.4	0.1486	0.1224	0.1832	0.1038	0.2225	0.0905	0.2559	0.0807	0.2738	1.4
1.5	0.1361	0.1124	0.1688	0.0958	0.2082	0.0840	0.2450	0.0753	0.2684	1.5
1.6	0.1243	0.1031	0.1546	0.0884	0.1933	0.0780	0.2325	0.0702	0.2610	1.6
1.7	0.1133	0.0944	0.1409	0.0815	0.1781	0.0723	0.2188	0.0655	0.2518	1.7
1.8	0.1029	0.0863	0.1278	0.0751	0.1628	0.0671	0.2041	0.0612	0.2408	1.8
1.9	0.0933	0.0789	0.1154	0.0692	0.1478	0.0623	0.1887	0.0571	0.2283	1.9
2.0	0.0845	0.0721	0.1038	0.0638	0.1333	0.0579	0.1729	0.0534	0.2145	2.0
2.1	0.0765	0.0660	0.0931	0.0589	0.1194	0.0538	0.1569	0.0499	0.1995	2.1
2.2	0.0692	0.0603	0.0833	0.0543	0.1064	0.0500	0.1412	0.0467	0.1838	2.2
2.3	0.0625	0.0551	0.0743	0.0501	0.0942	0.0465	0.1259	0.0437	0.1675	2.3
2.4	0.0565	0.0505	0.0663	0.0463	0.0831	0.0433	0.1112	0.0409	0.1510	2.4
2.5	0.0511	0.0462	0.0590	0.0428	0.0729	0.0403	0.0974	0.0383	0.1346	2.5
2.6	0.0463	0.0424	0.0526	0.0396	0.0639	0.0376	0.0847	0.0359	0.1186	2.6
2.7	0.0420	0.0389	0.0468	0.0367	0.0558	0.0351	0.0730	0.0337	0.1033	2.7
2.8	0.0381	0.0358	0.0418	0.0341	0.0486	0.0328	0.0625	0.0317	0.0888	2.8
2.9	0.0346	0.0329	0.0373	0.0316	0.0423	0.0306	0.0531	0.0298	0.0753	2.9
3.0	0.0315	0.0303	0.0333	0.0294	0.0369	0.0287	0.0450	0.0280	0.0631	3.0
3.1	0.0287	0.0280	0.0299	0.0274	0.0322	0.0268	0.0378	0.0264	0.0521	3.1
3.2	0.0262	0.0258	0.0268	0.0255	0.0281	0.0252	0.0318	0.0248	0.0424	3.2
3.3	0.0240	0.0239	0.0242	0.0238	0.0246	0.0236	0.0266	0.0234	0.0340	3.3
3.4	0.0220	0.0222	0.0218	0.0222	0.0216	0.0222	0.0223	0.0221	0.0269	3.4
3.5	0.0203	0.0206	0.0197	0.0208	0.0190	0.0209	0.0186	0.0209	0.0209	3.5
3.6	0.0186	0.0191	0.0179	0.0194	0.0168	0.0196	0.0157	0.0197	0.0160	3.6
3.7	0.0172	0.0178	0.0163	0.0182	0.0149	0.0185	0.0132	0.0187	0.0121	3.7
3.8	0.0159	0.0166	0.0149	0.0171	0.0133	0.0174	0.0112	0.0177	0.0089	3.8
3.9	0.0147	0.0155	0.0136	0.0161	0.0120	0.0165	0.0095	0.0167	0.0065	3.9
4.0	0.0137	0.0145	0.0125	0.0151	0.0107	0.0156	0.0082	0.0159	0.0047	4.0
4.1	0.0127	0.0136	0.0115	0.0142	0.0097	0.0147	0.0071	0.0151	0.0033	4.1
4.2	0.0118	0.0128	0.0106	0.0134	0.0088	0.0139	0.0062	0.0143	0.0023	4.2
4.3	0.0110	0.0120	0.0098	0.0127	0.0080	0.0132	0.0055	0.0136	0.0015	4.3
4.4	0.0103	0.0113	0.0091	0.0120	0.0073	0.0125	0.0048	0.0130	0.0010	4.4
4.5	0.0097	0.0106	0.0084	0.0113	0.0067	0.0119	0.0043	0.0123	0.0007	4.5
4.6	0.0091	0.0100	0.0078	0.0107	0.0062	0.0113	0.0039	0.0118	0.0004	4.6
4.7	0.0085	0.0094	0.0073	0.0102	0.0057	0.0107	0.0035	0.0112	0.0003	4.7
4.8	0.0080	0.0089	0.0068	0.0096	0.0053	0.0102	0.0032	0.0107	0.0002	4.8
4.9	0.0075	0.0084	0.0064	0.0092	0.0049	0.0097	0.0029	0.0102	0.0001	4.9
5.0	0.0071	0.0080	0.0060	0.0087	0.0046	0.0093	0.0027	0.0098		5.0
5.1	0.0067	0.0076	0.0056	0.0083	0.0043	0.0088	0.0025	0.0093		5.1
5.2	0.0063	0.0072	0.0053	0.0079	0.0040	0.0085	0.0023	0.0089		5.2
5.3	0.0060	0.0068	0.0050	0.0075	0.0037	0.0081	0.0021	0.0086		5.3
5.4	0.0057	0.0065	0.0047	0.0072	0.0035	0.0077	0.0020	0.0082		5.4
5.5	0.0054	0.0062	0.0045	0.0068	0.0033	0.0074	0.0018	0.0078		5.5
5.6	0.0051	0.0059	0.0042	0.0065	0.0031	0.0071	0.0017	0.0075		5.6
5.7	0.0049	0.0056	0.0040	0.0062	0.0029	0.0068	0.0016	0.0072		5.7
5.8	0.0046	0.0054	0.0038	0.0060	0.0028	0.0065	0.0015	0.0069		5.8
5.9	0.0044	0.0051	0.0036	0.0057	0.0026	0.0062	0.0014	0.0067		5.9
6.0	0.0042	0.0049	0.0034	0.0055	0.0025	0.0060	0.0014	0.0064		6.0

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.50

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
6.1	0.0040	0.0047	0.0033	0.0053	0.0024	0.0057	0.0013	0.0062		6.1
6.2	0.0038	0.0045	0.0031	0.0050	0.0022	0.0055	0.0012	0.0059		6.2
6.3	0.0037	0.0043	0.0030	0.0048	0.0021	0.0053	0.0011	0.0057		6.3
6.4	0.0035	0.0041	0.0028	0.0046	0.0020	0.0051	0.0011	0.0055		6.4
6.5	0.0034	0.0040	0.0027	0.0045	0.0019	0.0049	0.0010	0.0053		6.5
6.6	0.0032	0.0038	0.0026	0.0043	0.0018	0.0047	0.0010	0.0051		6.6
6.7	0.0031	0.0036	0.0025	0.0041	0.0018	0.0046	0.0009	0.0049		6.7
6.8	0.0030	0.0035	0.0024	0.0040	0.0017	0.0044	0.0009	0.0048		6.8
6.9	0.0029	0.0034	0.0023	0.0038	0.0016	0.0042	0.0009	0.0046		6.9
7.0	0.0027	0.0032	0.0022	0.0037	0.0015	0.0041	0.0008	0.0045		7.0
7.1	0.0026	0.0031	0.0021	0.0036	0.0015	0.0040	0.0008	0.0043		7.1
7.2	0.0025	0.0030	0.0020	0.0034	0.0014	0.0038	0.0007	0.0042		7.2
7.3	0.0024	0.0029	0.0019	0.0033	0.0014	0.0037	0.0007	0.0040		7.3
7.4	0.0023	0.0028	0.0019	0.0032	0.0013	0.0036	0.0007	0.0039		7.4
7.5	0.0023	0.0027	0.0018	0.0031	0.0012	0.0034	0.0007	0.0038		7.5
7.6	0.0022	0.0026	0.0017	0.0030	0.0012	0.0033	0.0006	0.0036		7.6
7.7	0.0021	0.0025	0.0017	0.0029	0.0012	0.0032	0.0006	0.0035		7.7
7.8	0.0020	0.0024	0.0016	0.0028	0.0011	0.0031	0.0006	0.0034		7.8
7.9	0.0020	0.0023	0.0015	0.0027	0.0011	0.0030	0.0006	0.0033		7.9
8.0	0.0019	0.0023	0.0015	0.0026	0.0010	0.0029	0.0005	0.0032		8.0
9.0	0.0014	0.0017	0.0011	0.0019	0.0007	0.0022	0.0004	0.0024		9.0
10.	0.0010	0.0013	0.0008	0.0015	0.0005	0.0017	0.0003	0.0019		10.
11.	0.0008	0.0010	0.0006	0.0012	0.0004	0.0013	0.0002	0.0015		11.
12.	0.0006	0.0008	0.0005	0.0009	0.0003	0.0011	0.0002	0.0012		12.
13.	0.0005	0.0006	0.0004	0.0008	0.0003	0.0009	0.0001	0.0010		13.
14.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0007		0.0008		14.
15.	0.0004	0.0004	0.0003	0.0005	0.0002	0.0006		0.0007		15.
16.	0.0003	0.0004	0.0002	0.0004	0.0001	0.0005		0.0006		16.
17.	0.0003	0.0003	0.0002	0.0004		0.0004		0.0005		17.
18.	0.0002	0.0003	0.0002	0.0003		0.0004		0.0004		18.
19.	0.0002	0.0002	0.0001	0.0003		0.0003		0.0004		19.
20.	0.0002	0.0002		0.0002		0.0003		0.0003		20.
21.	0.0001	0.0002		0.0002		0.0003		0.0003		21.
22.		0.0002		0.0002		0.0002		0.0003		22.
23.		0.0001		0.0002		0.0002		0.0002		23.
24.				0.0002		0.0002		0.0002		24.
25.				0.0001		0.0002		0.0002		25.
26.						0.0001		0.0002		26.
27.								0.0001		27.

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
0.0	0.2835	0.2821	0.2821	0.2782	0.2782	0.2720	0.2720	0.2642	0.2642	0.0
0.1	0.2824	0.2793	0.2833	0.2736	0.2813	0.2660	0.2768	0.2569	0.2703	0.1
0.2	0.2800	0.2750	0.2828	0.2677	0.2828	0.2587	0.2801	0.2486	0.2752	0.2
0.3	0.2761	0.2692	0.2807	0.2604	0.2827	0.2503	0.2819	0.2395	0.2787	0.3
0.4	0.2706	0.2621	0.2770	0.2520	0.2809	0.2410	0.2821	0.2297	0.2808	0.4
0.5	0.2636	0.2537	0.2717	0.2427	0.2775	0.2310	0.2807	0.2193	0.2813	0.5
0.6	0.2553	0.2443	0.2649	0.2325	0.2725	0.2204	0.2776	0.2086	0.2802	0.6
0.7	0.2459	0.2340	0.2568	0.2216	0.2660	0.2094	0.2730	0.1976	0.2775	0.7
0.8	0.2355	0.2229	0.2474	0.2103	0.2581	0.1981	0.2668	0.1865	0.2731	0.8
0.9	0.2244	0.2114	0.2371	0.1986	0.2489	0.1866	0.2591	0.1754	0.2672	0.9
1.0	0.2126	0.1994	0.2258	0.1868	0.2385	0.1751	0.2501	0.1644	0.2599	1.0
1.1	0.2003	0.1872	0.2138	0.1750	0.2272	0.1637	0.2399	0.1536	0.2511	1.1
1.2	0.1878	0.1750	0.2013	0.1632	0.2152	0.1526	0.2286	0.1431	0.2411	1.2
1.3	0.1752	0.1628	0.1885	0.1516	0.2025	0.1417	0.2165	0.1330	0.2299	1.3
1.4	0.1626	0.1508	0.1755	0.1404	0.1894	0.1312	0.2037	0.1232	0.2178	1.4
1.5	0.1502	0.1392	0.1626	0.1295	0.1761	0.1212	0.1904	0.1139	0.2050	1.5
1.6	0.1381	0.1279	0.1498	0.1191	0.1628	0.1116	0.1768	0.1051	0.1915	1.6
1.7	0.1265	0.1172	0.1373	0.1092	0.1495	0.1025	0.1632	0.0968	0.1777	1.7
1.8	0.1153	0.1070	0.1252	0.0999	0.1366	0.0940	0.1495	0.0890	0.1637	1.8
1.9	0.1047	0.0973	0.1136	0.0911	0.1241	0.0860	0.1362	0.0817	0.1498	1.9
2.0	0.0948	0.0883	0.1026	0.0830	0.1121	0.0785	0.1232	0.0749	0.1360	2.0
2.1	0.0855	0.0799	0.0923	0.0754	0.1007	0.0716	0.1107	0.0685	0.1225	2.1
2.2	0.0768	0.0721	0.0827	0.0683	0.0899	0.0653	0.0989	0.0627	0.1096	2.2
2.3	0.0689	0.0650	0.0738	0.0619	0.0800	0.0594	0.0877	0.0573	0.0973	2.3
2.4	0.0616	0.0585	0.0656	0.0560	0.0707	0.0540	0.0773	0.0524	0.0857	2.4
2.5	0.0549	0.0525	0.0581	0.0506	0.0623	0.0491	0.0678	0.0478	0.0749	2.5
2.6	0.0489	0.0471	0.0514	0.0457	0.0546	0.0446	0.0590	0.0437	0.0649	2.6
2.7	0.0435	0.0422	0.0453	0.0413	0.0477	0.0405	0.0511	0.0399	0.0558	2.7
2.8	0.0386	0.0378	0.0398	0.0372	0.0415	0.0368	0.0440	0.0364	0.0477	2.8
2.9	0.0343	0.0339	0.0350	0.0336	0.0360	0.0334	0.0377	0.0333	0.0403	2.9
3.0	0.0304	0.0303	0.0307	0.0303	0.0312	0.0304	0.0321	0.0305	0.0338	3.0
3.1	0.0270	0.0272	0.0269	0.0274	0.0269	0.0276	0.0273	0.0279	0.0282	3.1
3.2	0.0239	0.0244	0.0235	0.0248	0.0232	0.0252	0.0230	0.0255	0.0232	3.2
3.3	0.0213	0.0219	0.0206	0.0224	0.0199	0.0229	0.0194	0.0234	0.0190	3.3
3.4	0.0189	0.0197	0.0181	0.0203	0.0172	0.0209	0.0163	0.0215	0.0154	3.4
3.5	0.0168	0.0177	0.0159	0.0185	0.0148	0.0191	0.0136	0.0197	0.0124	3.5
3.6	0.0150	0.0159	0.0139	0.0168	0.0127	0.0175	0.0113	0.0181	0.0099	3.6
3.7	0.0134	0.0144	0.0123	0.0153	0.0110	0.0160	0.0095	0.0167	0.0078	3.7
3.8	0.0120	0.0130	0.0108	0.0139	0.0095	0.0147	0.0079	0.0154	0.0061	3.8
3.9	0.0108	0.0118	0.0096	0.0127	0.0082	0.0135	0.0066	0.0142	0.0047	3.9
4.0	0.0097	0.0107	0.0085	0.0116	0.0071	0.0124	0.0055	0.0131	0.0036	4.0
4.1	0.0087	0.0098	0.0076	0.0106	0.0062	0.0114	0.0046	0.0121	0.0028	4.1
4.2	0.0079	0.0089	0.0068	0.0098	0.0055	0.0106	0.0039	0.0113	0.0021	4.2
4.3	0.0072	0.0081	0.0061	0.0090	0.0048	0.0098	0.0033	0.0104	0.0016	4.3
4.4	0.0065	0.0075	0.0055	0.0083	0.0042	0.0090	0.0028	0.0097	0.0012	4.4
4.5	0.0060	0.0069	0.0049	0.0077	0.0038	0.0084	0.0024	0.0090	0.0008	4.5
4.6	0.0054	0.0063	0.0045	0.0071	0.0034	0.0078	0.0021	0.0084	0.0006	4.6
4.7	0.0050	0.0058	0.0041	0.0066	0.0030	0.0072	0.0018	0.0079	0.0004	4.7
4.8	0.0046	0.0054	0.0037	0.0061	0.0027	0.0068	0.0016	0.0073	0.0003	4.8
4.9	0.0042	0.0050	0.0034	0.0057	0.0025	0.0063	0.0014	0.0069	0.0002	4.9
5.0	0.0039	0.0046	0.0031	0.0053	0.0023	0.0059	0.0013	0.0064	0.0001	5.0

PROBABILITY DENSITY FUNCTION P(X|ALPHA,BETA)

ALPHA = 1.75

X	BETA = .00	-.25	.25	-.50	.50	-.75	.75	-1.00	1.00	X
5.1	0.0036	0.0043	0.0029	0.0050	0.0021	0.0055	0.0011	0.0060	0.0000	5.1
5.2	0.0034	0.0040	0.0027	0.0046	0.0019	0.0052	0.0010	0.0057		5.2
5.3	0.0031	0.0038	0.0025	0.0043	0.0017	0.0049	0.0009	0.0054		5.3
5.4	0.0029	0.0035	0.0023	0.0041	0.0016	0.0046	0.0008	0.0050		5.4
5.5	0.0027	0.0033	0.0021	0.0038	0.0015	0.0043	0.0008	0.0048		5.5
5.6	0.0026	0.0031	0.0020	0.0036	0.0014	0.0041	0.0007	0.0045		5.6
5.7	0.0024	0.0029	0.0019	0.0034	0.0013	0.0038	0.0007	0.0042		5.7
5.8	0.0023	0.0027	0.0017	0.0032	0.0012	0.0036	0.0006	0.0040		5.8
5.9	0.0021	0.0026	0.0016	0.0030	0.0011	0.0034	0.0006	0.0038		5.9
6.0	0.0020	0.0024	0.0015	0.0029	0.0011	0.0032	0.0005	0.0036		6.0
6.1	0.0019	0.0023	0.0015	0.0027	0.0010	0.0031	0.0005	0.0034		6.1
6.2	0.0018	0.0022	0.0014	0.0026	0.0009	0.0029	0.0005	0.0033		6.2
6.3	0.0017	0.0021	0.0013	0.0024	0.0009	0.0028	0.0005	0.0031		6.3
6.4	0.0016	0.0020	0.0012	0.0023	0.0008	0.0026	0.0004	0.0029		6.4
6.5	0.0015	0.0019	0.0012	0.0022	0.0008	0.0025	0.0004	0.0028		6.5
6.6	0.0014	0.0018	0.0011	0.0021	0.0007	0.0024	0.0004	0.0027		6.6
6.7	0.0014	0.0017	0.0010	0.0020	0.0007	0.0023	0.0004	0.0026		6.7
6.8	0.0013	0.0016	0.0010	0.0019	0.0007	0.0022	0.0003	0.0024		6.8
6.9	0.0012	0.0015	0.0009	0.0018	0.0006	0.0021	0.0003	0.0023		6.9
7.0	0.0012	0.0015	0.0009	0.0017	0.0006	0.0020	0.0003	0.0022		7.0
7.1	0.0011	0.0014	0.0009	0.0017	0.0006	0.0019	0.0003	0.0021		7.1
7.2	0.0011	0.0013	0.0008	0.0016	0.0006	0.0018	0.0003	0.0020		7.2
7.3	0.0010	0.0013	0.0008	0.0015	0.0005	0.0017	0.0003	0.0020		7.3
7.4	0.0010	0.0012	0.0008	0.0014	0.0005	0.0017	0.0003	0.0019		7.4
7.5	0.0009	0.0012	0.0007	0.0014	0.0005	0.0016	0.0002	0.0018		7.5
7.6	0.0009	0.0011	0.0007	0.0013	0.0005	0.0015	0.0002	0.0017		7.6
7.7	0.0009	0.0011	0.0007	0.0013	0.0005	0.0015	0.0002	0.0017		7.7
7.8	0.0008	0.0010	0.0006	0.0012	0.0004	0.0014	0.0002	0.0016		7.8
7.9	0.0008	0.0010	0.0006	0.0012	0.0004	0.0014	0.0002	0.0015		7.9
8.0	0.0008	0.0010	0.0006	0.0011	0.0004	0.0013	0.0002	0.0015		8.0
9.	0.0005	0.0007	0.0004	0.0008	0.0003	0.0009	0.0001	0.0010		9.
10.	0.0004	0.0005	0.0003	0.0006	0.0002	0.0007		0.0008		10.
11.	0.0003	0.0004	0.0002	0.0004	0.0001	0.0005		0.0006		11.
12.	0.0002	0.0003	0.0002	0.0003		0.0004		0.0005		12.
13.	0.0002	0.0002	0.0001	0.0003		0.0003		0.0004		13.
14.	0.0001	0.0002		0.0002		0.0002		0.0003		14.
15.		0.0001		0.0002		0.0002		0.0002		15.
16.				0.0001		0.0002		0.0002		16.
17.						0.0001		0.0002		17.
18.								0.0001		18.

P(X|2,BETA)

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.28209	0.28139	0.27929	0.27582	0.27103	0.26500	0.25782	0.24957	0.24039	0.23038
1.	0.21970	0.20846	0.19681	0.18489	0.17282	0.16073	0.14875	0.13697	0.12549	0.11440
2.	0.10378	0.09367	0.08412	0.07517	0.06684	0.05913	0.05205	0.04559	0.03974	0.03446
3.	0.02973	0.02553	0.02181	0.01854	0.01568	0.01319	0.01105	0.00920	0.00763	0.00629
4.	0.00517	0.00422	0.00343	0.00277	0.00223	0.00179	0.00142	0.00113	0.00089	0.00070
5.	0.00054	0.00042	0.00033	0.00025	0.00019	0.00015	0.00011	0.00008	0.00006	0.00005
6.	0.00003	0.00003	0.00002	0.00001	0.00001					

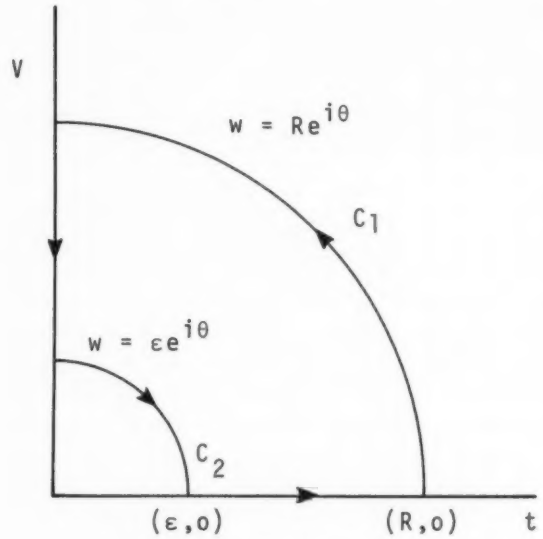

FIGURE 1. *Contour for* $p_x(x; \alpha, \beta)$.

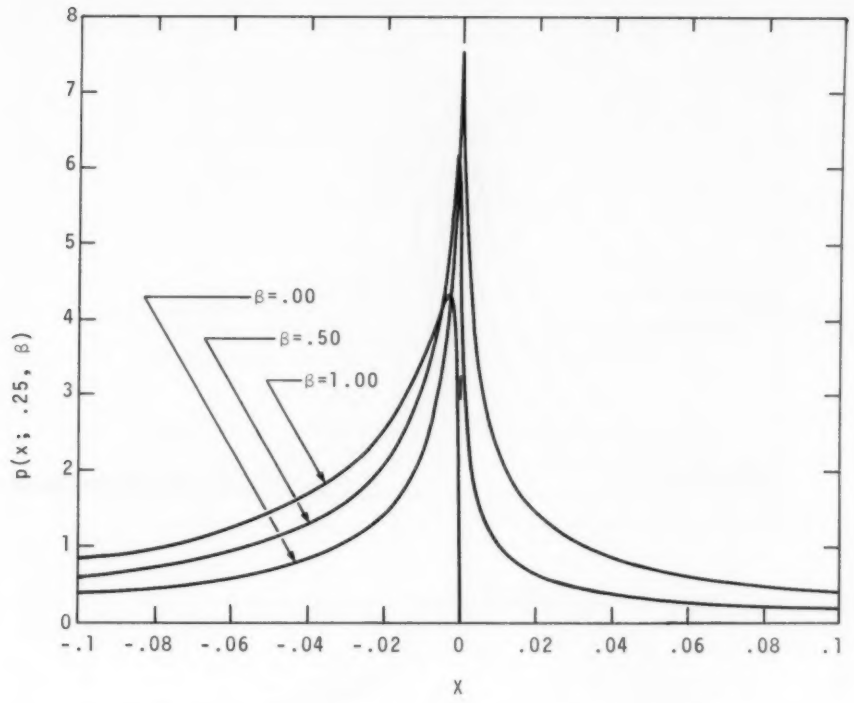

FIGURE 2. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 0.25$.

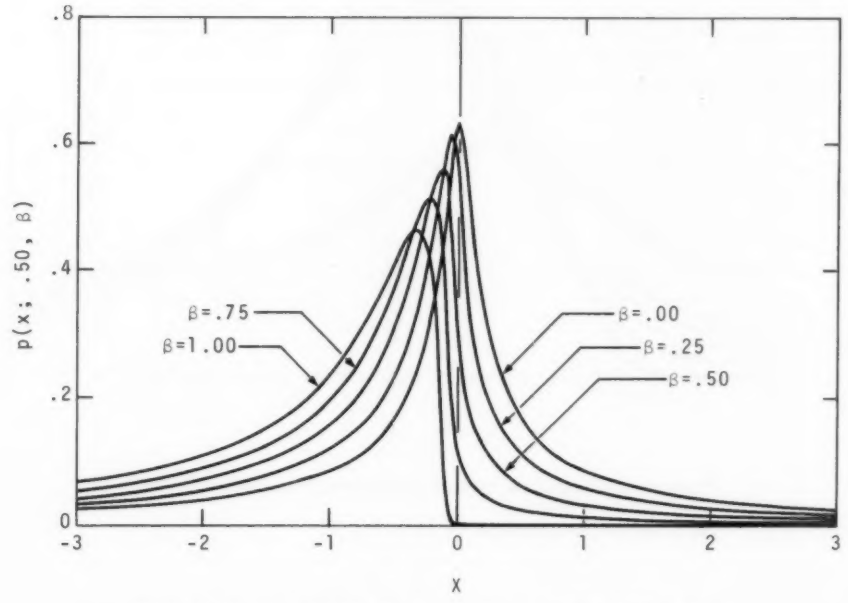

FIGURE 3. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 0.50$.

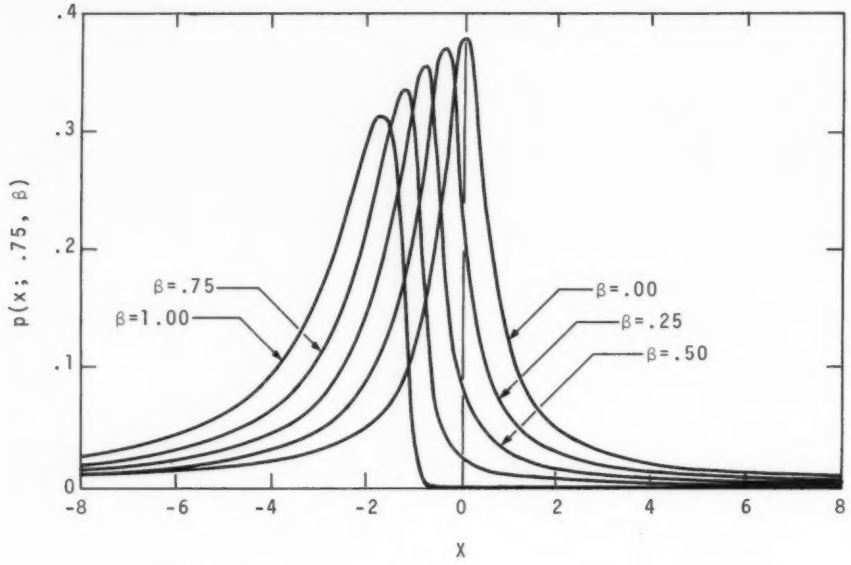

FIGURE 4. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 0.75$.

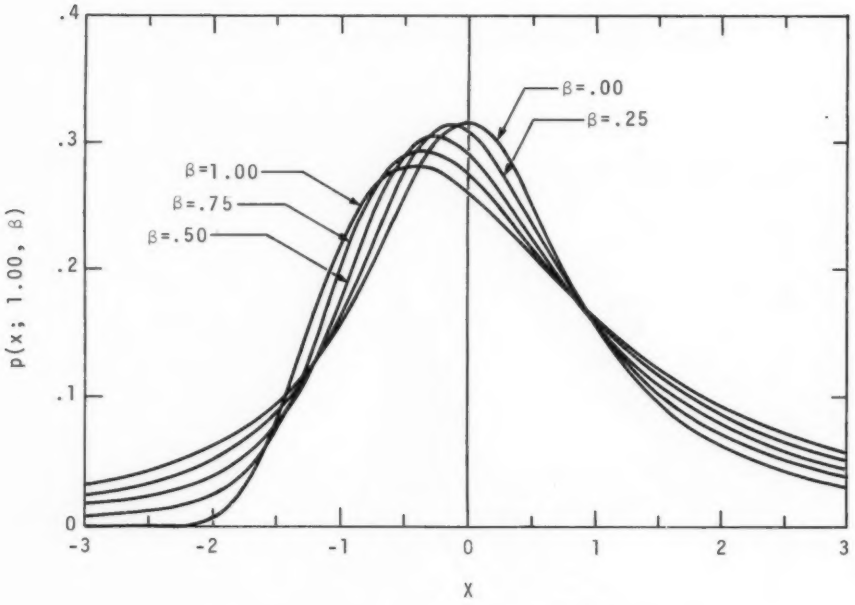

FIGURE 5. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 1.00$.

Note reversal of location of heavier tail from that for $\alpha \neq 1$.

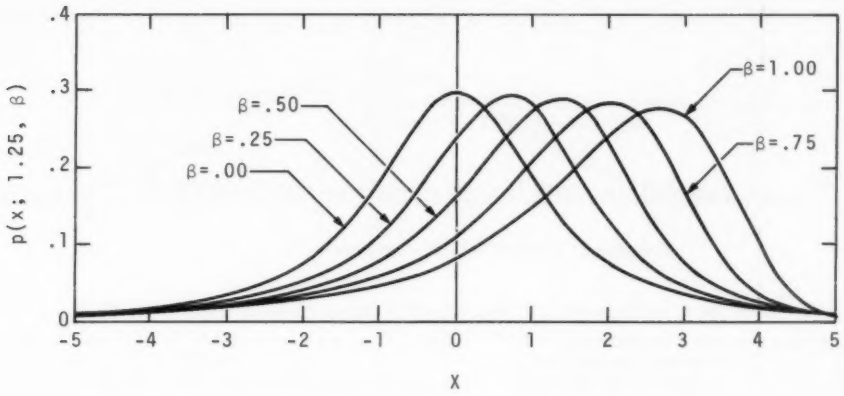

FIGURE 6. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 1.25$.

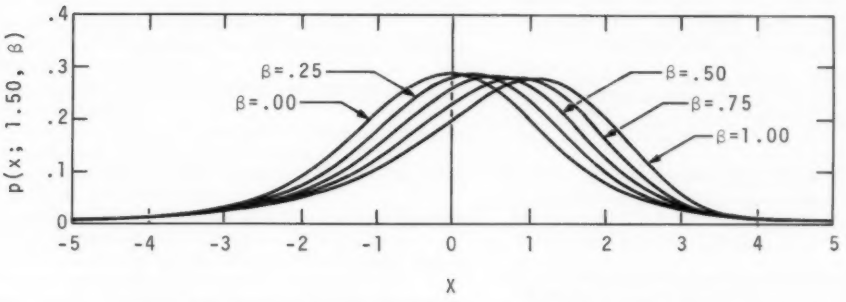

FIGURE 7. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 1.50$.

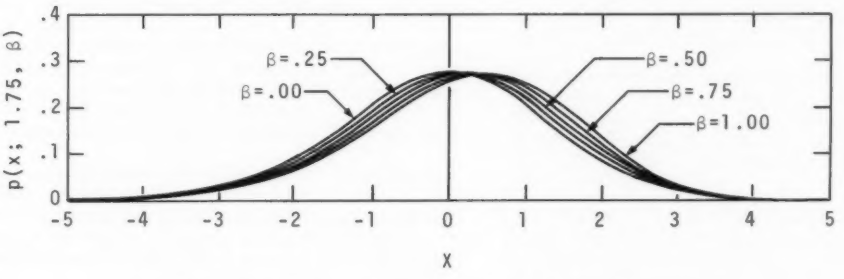

FIGURE 8. *Stable probability density functions* $p(x; \alpha, \beta)$ *for* $\alpha = 1.75$.

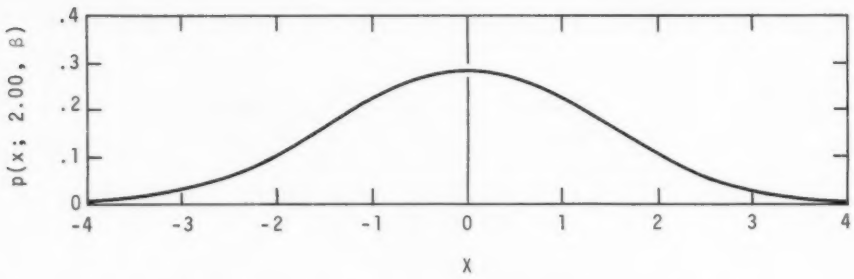

FIGURE 9. *Stable probability density function* p(x; α, β) *for* $\alpha=2$ (*the normal* pdf).

(Paper 77B3&4–390)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematical Sciences
Vol. 77B, Nos. 3 and 4, July-December 1973

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Methods for calculating parameters for quadrature formulas of the type defined by N. M. Korobov and E. Hlawka have been proposed by several authors. In this paper a number of sequences of such parameter sets are found, by methods due to L. K. Hua and Y. Wang, and to N. M. Korobov. An error bound for the quadrature formula corresponding to each parameter set is found, by a modification of a method of Korobov. Conjectures about the asymptotic properties of these error bounds are formulated. Practical questions about the various methods of calculating parameters are discussed.

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Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X, with S_0 and S_2 closed and S_1 open in S_2. If f is a compact mapping of S_2 into X such that $\cup_{j=1}^{m} f^j(S_1) \subset S_2$ and $f^m(S_1) \cup f^{m+1}(S_1) \subset S_0$ for some $m > 0$, then f has a fixed point in S_0. (This extends a result of F. E. Browder published in 1959.) Also, if $\{T_t : t \in R^+\}$ is a continuous flow on the Banach space X, $S_0 \subset S_1 \subset S_2$ are convex subsets of X with S_0 and S_2 compact and S_1 open in S_2, and $T_{t_0}(S_1) \subset S_0$ for some $t_0 > 0$, where $T_t(S_1) \subset S_2$ for all $t \leqq t_0$, then there exists $x_0 \in S_0$ such that $T_t(x_0) = x_0$ for all $t \geqq 0$. Minor extensions of Browder's work on "nonejective" and "nonrepulsive" fixed points are also given, with similar results for flows.

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JOURNAL OF RESEARCH of the National Bureau of Standards—Mathematical Sciences
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Library of Congress Catalog Card Number: 62–4414

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